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Learning in Advance Selling with Heterogeneous Consumers^{*}

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Abstract

The advance selling strategy is implemented when a firm offers consumers the opportunity to order its product in advance of the regular selling season. Advance selling reduces uncertainty for both the firm and the buyer and enables the firm to update its forecast of future demand. The distinctive feature of the present study of advance selling is that we divide consumers into two groups, experienced and inexperienced. Experienced consumers know their valuations of the product in advance. The presence of experienced consumers yields new insights. Specifically, pre-orders from experienced consumers lead to a more precise forecast of future demand by the firm. We show that the firm will always adopt advance selling and that the optimal pre-order price may be at a discount or a premium relative to the regular selling price.

Key words: advance selling, the Newsvendor Problem, demand uncertainty, experienced consumers, inexperienced consumers.

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1 Introduction

Advance selling occurs when firms and retailers offer consumers the opportunity to order a product or service in advance of the regular selling season. Remarkable developments in the Internet and information technology have made advance selling an economically efficient strategy in many product categories.¹ Examples include new books, movies and CDs, software, electronic games, smart phones, travel services and vacation packages.

There are several major advantages of advance selling. First, advance selling reduces uncertainty for both the firm and the buyer, because advance orders are pre-committed. In situations where the firm needs to decide how much to produce (procure) prior to the regular selling season, advance orders reduce demand uncertainty. For the buyer, an advance order guarantees delivery of the product in the regular selling season, possibly at a discount to the retail price. Second, orders from advance selling may provide valuable information for the firm to better forecast the future demand. In particular, the firm may be able to update its forecast of the size of consumer pool and the distribution of consumers' valuations. Finally, advance selling may increase the overall demand. Indeed, when a consumer pre-orders the product, she commits to purchase it. In the absence of advance selling the same consumer will not purchase the product if she learns later her valuation is low.²

The motivation for the present study is based on two observations. First, some products were not made available for pre-orders when they were first introduced, but pre-orders became possible for later versions, with many experienced consumers placing their orders in advance of the regular selling season. This observation points to an important role played by experienced consumers. Table 1 reports the product release history of several well-known products.³ Second, although advance selling discounts are often practiced, some retailers set high prices for pre-orders and then cut their prices after the product release date. For highly sought-after products, some consumers (new technology lovers or loyal fans) are willing to pay a premium price for guaranteed delivery on the release date.

While inexperienced consumers learn more about their valuations of the product when it becomes available, experienced consumers are likely to have a good idea about their valuations of the product in advance. Therefore, experienced consumers have less incentives to wait until the regular selling season. It follows that when there are experienced consumers, advance selling is more likely to be utilized by consumers. In addition, pre-orders from experienced consumers are more informative than those from inexperienced consumers. One can thus conclude that the presence of experienced consumers makes the first two of the aforementioned advantages of advance selling more pronounced.

A number of papers in the literature have emphasized some or all of the three advantages of advance selling (see the literature review in Section 2), but none have modeled experienced consumers and the role they play in advance selling. This paper is the first study of advance selling with both experienced consumers and inexperienced consumers. Moreover, our paper yields the result that both advance selling at a discount and at a premium can arise in equilibrium.

¹See empirical studies by Moe and Fader (2002, 2009), Hui, Eliashberg, and George (2008), and Sainam, Balasubramanian, and Bayus (2010).

²Similar points have been made by other authors, e.g., Xie and Shugan (2001).

³This table is compiled by the authors based on various Internet publications.

Table 1: Release history and pre-order availability for several products

Product	Version	Release date	Pre-order availability
Amazon Kindle	Kindle	Nov. 19, 2007	No
	Kindle 2	Feb. 23, 2009	Yes
	Kindle 3	Aug. 27, 2010	Yes
	Kindle 4	Sept. 27, 2011	Yes
Harry Potter	Book 1	Sep. 1, 1998	No
	Book 2	Jun. 2, 1999	No
	Book 3	Sep. 8, 1999	Yes
	Book 4	Jul. 8, 2000	Yes
	Book 5	Jun. 21, 2003	Yes
	Book 6	Jul. 16, 2005	Yes
	Book 7	Jul. 21, 2007	Yes
iPhone	iPhone	Jun. 29, 2007	No
	iPhone 3G	Jul. 11, 2008	No
	iPhone 3GS	Jun. 19, 2009	Yes
	iPhone 4	Jun. 24, 2010	Yes
	iPhone 4S	Oct. 14, 2011	Yes
	iPhone 5	Sept. 21, 2012	Yes
iPod Touch	iPod Touch 1st	Sep. 5, 2007	No
	iPod Touch 2nd	Sep. 9, 2008	No
	iPod Touch 3rd	Sep. 9, 2009	Yes
	iPod Touch 4th	Sep. 1, 2010	Yes

Our model has two periods. The first is the advance selling season and the second is the regular selling season. In the first period, the firm chooses whether to make its product available for pre-orders, and if so, the pre-order price. There are two groups of consumers – experienced and inexperienced. Experienced consumers know their valuations of the product from the outset, while inexperienced consumers learn their valuations only in the second period. All consumers decide whether to pre-order the product (if this option is available) or wait until the regular selling season, in which they will face a probability of not being able to get the product (the stock-out probability). At the conclusion of the first period, the firm must choose its production quantity, which has to be at least the size of pre-orders. The product is delivered at the end of the second period.

Consumers are heterogeneous in their valuations, which are assumed to follow a normal distribution. The firm does not know the mean of this distribution. The group size of experienced consumers is fixed and known to the firm. However, the firm is uncertain about the number of

inexperienced consumers.

In the second period the firm faces the Newsvendor Problem by analogy with the situation faced by a newsvendor who must decide how many copies of the day's paper to stock on a newsstand before observing demand, knowing that unsold copies will become worthless by the end of the day. If the produced quantity is greater than the realized demand, the firm must dispose of the remaining units at a loss (due to the salvage value being below the marginal cost). If the produced quantity is lower than the realized demand, the firm forgoes some profit.⁴

Our main research questions are the following. Will the firm adopt the advance selling strategy? If so, will an advance selling discount or premium be offered? How do experienced and inexperienced consumers behave in the advance selling season? What can the firm learn from pre-orders? How much should the firm produce? How are the answers to (some of) these questions affected by parameters of the model, such as the salvage value, the demand uncertainty, the variation of consumer valuations, and the composition of experienced/inexperienced consumers in the population?

Our main results are summarized below.

- For the firm, advance selling is strictly better than no advance selling. The firm has three possible pricing strategies for pre-orders: advance selling at the regular selling price, advance selling at a discount price, and advance selling at a premium price.
- Under advance selling at the regular selling price, experienced consumers do not wait until the regular selling season. Those with valuations above the advance selling price pre-order. All inexperienced consumers wait until the regular selling season.
- The firm's optimal advance selling discount price is such that no consumer waits until the regular selling season. Specifically, experienced consumers whose valuations are above the advance selling price and all inexperienced consumers pre-order.
- Under advance selling at a premium, inexperienced consumers do not pre-order. For experienced consumers, those with high valuations pre-order, those with valuations below the regular selling price never buy the product, and the rest wait to purchase in the regular selling season.
- The firm learns from pre-orders. It learns whether there are any consumers who have chosen to wait until the regular selling season. If nobody waits, the firm only needs to fill all pre-orders. In the case when some consumers wait, the firm learns the mean of consumers' valuations.
- Our sensitivity analysis in regard to changes in some parameters of the model yields several interesting results, some intuitive and some counterintuitive. For example, as the salvage value decreases, the firm's expected profit may decrease (intuitive), but may also increase (counterintuitive). Likewise, as the proportion of experienced consumers decreases, the firm's expected profit may decrease (intuitive), but may also increase (counterintuitive).

⁴If there is no uncertainty about the second-period demand or if the salvage value equals the marginal cost, then the Newsvendor Problem disappears.

Our paper contains several contributions to the literature on advance selling. First, as mentioned earlier, we study advance selling in a model with inexperienced as well as experienced consumers. We believe our model captures an important aspect of the advance selling phenomena. Second, learning by the firm in our model is not only on the consumer pool but also on the distribution of consumers' valuations of the product. Finally, like in the recent working papers by Li and Zhang (2012) and Zhao and Pang (2011), the stock-out probability that consumers face when they wait until the regular selling season is endogenously determined in our model. In most of the literature, the stock-out probability has been modeled as exogenously given. We think that the correct way to model the stock-out probability is through endogenous determination, since this probability affects consumers' choices in the advance selling season and these choices in turn affect the stock-out probability in the regular selling season.

After the literature review (Section 2), the rest of the paper is organized as follows. In Section 3 we introduce the model. In Section 4 we derive consumers' optimal responses and the retailer's equilibrium strategy. In Section 5 sensitivity analysis results are presented. In Section 6 we explore the possibility of advance selling at a price premium. Concluding remarks are provided in Section 7. Proofs of all lemmas and propositions, as well as derivations for some expressions and claims, are relegated to the Appendix.

2 Literature Review

Several strands of the literature have studied advance selling. One deals with advance selling from manufacturers to retailers, e.g., Cachon (2004) and Özer and Wei (2006). Another is on advance selling from firms and retailers to consumers under limited capacity, with applications to the airline and hotel industries (Gale and Holmes, 1993, Dana, 1998). The literature that is closest to the present study is on advance selling from firms and retailers to consumers without capacity constraints.

Our review below focuses on the third strand.⁵ Two modeling approaches have been adopted by researchers. In the first approach consumers are non-strategic in their decisions on whether to pre-order the product. In the second approach consumers are strategic.

Papers that model consumers as non-strategic include Weng and Parlar (1999), Tang, Rajaram, Alptekinoglu, and Ou (2004), McCardle, Rajaram, and Tang (2004), and Chen and Parlar (2005). In all of these papers the fraction of consumers who place advance orders is an exogenously given decreasing function of the advance selling price.⁶ In Tang, Rajaram, Alptekinoglu, and Ou (2004) and McCardle, Rajaram, and Tang (2004) there are two brands belonging to rival firms. Advance selling by a firm attracts customers of the other brand. The former paper examines the decision on advance selling by a single firm, while the latter focuses on competition between two firms in adopting the advance selling strategy. Another paper is Boyaci and Özer (2010) that considers advance selling in multiple periods and learning by the firm about current and future demands.

⁵The main difference between the second and the third strands is that in situations of limited capacity firms mainly choose prices, while without capacity constraints firms choose their production quantities as well as prices.

⁶In Chen and Parlar (2005) an alternative model is considered, in which the probability that each consumer orders in advance is a beta-distributed random variable.

Several papers have treated consumers as strategic. Strategic consumers compare the options of ordering in advance and of waiting until the regular selling season. In the model by Xie and Shugan (2001) consumers are uncertain about their valuations in the advance selling season. This uncertainty makes advance selling profitable when it induces consumers who would otherwise not purchase to pre-order. Chu and Zhang (2011) allow the firm to control the release of information about the product at pre-order. They show that the seller may want to release some information or none, but never all. While in Xie and Shugan (2001) and Chu and Zhang (2011) consumers are homogeneous in the advance selling season (before their valuation uncertainty is resolved), in the studies cited below they are heterogeneous. Zhao and Steckle (2010) classify consumers according to whether they are loss averse. A loss averse consumer is more averse to a negative surplus (when the realized valuation is below the advance selling price) than is attracted to the equivalent positive surplus. Prasad, Steckle, and Zhao (2011) and Zhao and Pang (2011) divide consumers into two groups. The informed group consists of consumers who know about the option to buy in advance, while the uninformed group is not aware of this option. In Nocke, Peitz, and Rosar (2011) consumers differ in their expected valuations. In contrast to the above papers, Li and Zhang (2012) assume deterministic consumer valuations. Consumers with high valuations arrive in the advance selling season, consumers with low valuations arrive in the regular selling season.⁷

The major issues studied in this literature are (i) the Newsvendor Problem, and (ii) learning and updating by the firm about consumer demand. The Newsvendor Problem arises because the firm, facing uncertain demand, has to choose its production quantity prior to the regular selling season. Obviously, learning from pre-orders benefits the firm because it helps to better forecast the demand in the regular selling season. Both issues are also central in our paper. Because we assume that the mean of the distribution of consumers' valuations is unknown to the retailer, learning in our model is not only on the consumer pool, but also on the distribution of consumers' valuations.

Consumers in our model are strategic. The key difference between our paper and extant literature on advance selling with strategic consumers is the introduction of experienced consumers into the model. As stated before, experienced consumers make the strategy of advance selling more attractive to the firm.

3 Model Setup

Consider a firm or a retailer who sells a product over two periods.⁸ The first period is the advance selling season and the second period is the regular selling season. Any consumer who pre-orders in the first period is guaranteed delivery of the product in the second period. Those who do not pre-order can buy in the regular selling season, but there is a risk that the product will be out of stock. There are two groups of consumers – experienced and inexperienced. Experienced consumers are assumed to know their valuations of the product from the outset, whereas

⁷Xie and Shugan (2001), Li and Zhang (2012), and Zhao and Pang (2011) will be further discussed in Section 6 that considers advance selling at a price premium.

⁸From now on we shall speak of the firm as the retailer.

Table 2: Notation

Parameters	
c	marginal cost
s	salvage value
p	price in the regular selling season
m_e	number of experienced consumers
Random variables	
$v \mu \sim N(\mu, \sigma^2)$	distribution of consumers' valuations, cdf $F_{v \mu}(v) = \Phi\left(\frac{v-\mu}{\sigma}\right)$
$\mu \sim N(\mu_0, \psi^2)$	distribution of μ , cdf $F_\mu(\mu) = \Phi\left(\frac{\mu-\mu_0}{\psi}\right)$
$v \sim N(\mu_0, \sigma^2 + \psi^2)$	unconditional distribution of consumers' valuations, cdf $F_v(v) = \Phi\left(\frac{v-\mu_0}{\sqrt{\sigma^2 + \psi^2}}\right)$
$M_i \sim \text{LN}(\nu_i, \tau_i^2)$	number of inexperienced consumers, cdf $G(M_i) = \Phi\left(\frac{\ln M_i - \nu_i}{\tau_i}\right)$ and mean $m_i = \exp\left\{\nu_i + \frac{\tau_i^2}{2}\right\}$
Decision variables	
q	quantity produced for the regular selling season
Q	total quantity produced (including pre-orders)
x	advance selling price
Other notation	
D_1, D_2	demands in the advance and regular selling seasons
π	retailer's expected profit from the regular selling season
Π	retailer's total expected profit (includes pre-orders)
η	stock-out probability
β	$= \frac{p-c}{p-s}$
$\bar{F}$	$= 1 - F$

inexperienced consumers learn their valuations only in the second period.⁹ Each consumer is willing to buy at most one unit of the product.

The number of experienced consumers is exogenously given, denoted by m_e .¹⁰ The number of inexperienced consumers, M_i is a random variable; the distribution of M_i is lognormal $\text{LN}(\nu_i, \tau_i^2)$ with the mean $m_i = \exp\left\{\nu_i + \frac{\tau_i^2}{2}\right\}$.¹¹ Both m_e and the distribution of M_i are

⁹ As discussed in the Introduction, experienced consumers maybe taken as those who have purchased earlier versions of the product in the past, so it is reasonable to assume that they know their willingness to pay for the new version.

¹⁰ The non-randomness of m_e follows from footnote 9.

¹¹ We use a lognormal distribution to avoid negative realizations of the number of inexperienced consumers.

common knowledge.

Consumers' valuations of the product are i.i.d. normal with mean μ and variance σ^2 .¹² The retailer and consumers do not know μ . We model this uncertainty by assuming that μ is drawn from the normal distribution with mean μ_0 and variance ψ^2 . That is,

$$v|\mu \sim N(\mu, \sigma^2)$$

and

$$\mu \sim N(\mu_0, \psi^2).$$

In the Appendix we show that the unconditional distribution of v is

$$v \sim N(\mu_0, \sigma^2 + \psi^2). \quad (1)$$

We denote by $F_{v|\mu}(v)$, $f_{v|\mu}(v)$, $F_\mu(\mu)$, $f_\mu(\mu)$, $F_v(v)$, and $f_v(v)$ the corresponding cumulative distribution and density functions.

The marginal production cost is c and the price during the regular selling season is p .¹³ For each unsold unit of the product at the end of the regular selling season the retailer gets its salvage value s . We assume $s < c < p$.

The retailer decides on the advance selling price x in the beginning of the advance selling season. After the conclusion of the advance selling season the retailer must decide how much to produce. Let D_1 denote the number of consumers who buy in the advance selling season. Then the retailer's quantity choice is $Q = D_1 + q$, where D_1 fulfills the pre-orders. Quantity q satisfies the (stochastic) demand during the regular selling season, denoted by D_2 .

Table 2 lists the notation introduced above and also some of the notation introduced later. Figure 1 displays the timeline of the model. In the beginning of the first period, Nature determines μ and all experienced consumers learn their valuations. The retailer announces the advance selling price x (together with the regular selling price p). Each consumer then decides whether to pre-order. At the end of the first period, the retailer observes the number of pre-orders D_1 , updates his forecast of the second-period demand D_2 and chooses his production quantity Q . During the second period, all inexperienced consumers learn their valuations and those consumers who did not pre-order then decide whether to purchase the product at price p . The product is delivered at the end of the second period.

4 Equilibrium Analysis

Our goal in this section is to find the optimal advance selling price, assuming that $x \leq p$ so as to focus on pre-order discounting. To do this, we first derive consumers' optimal responses to any

¹²The two groups are assumed to have the same distribution of valuations. Some of the experienced consumers have realized high valuations from purchasing earlier versions of the product in the past, others low valuations. So, their valuation distribution is not necessarily different from that of inexperienced consumers. On the other hand, one may argue that the valuation distributions are different between experienced and inexperienced consumers. Section 7 (Concluding Remarks) has further discussion of this assumption.

¹³In our model p is exogenously given. In Prasad, Stecke, and Zhao (2010) and Zhao and Stecke (2010), p is called the market (spot) price. Among the papers reviewed in Section 2, about half treat the regular selling season price as exogenous. In the Concluding Remarks section we have a discussion on endogenous p .

Figure 1: Timeline of the model

advance selling price, and how the retailer learns from pre-orders and chooses his production quantity. We then determine the endogenous stock-out probability and present the retailer's expected profit function.

4.1 Consumers' Optimal Purchasing Decisions

Since experienced consumers know their valuations from the outset, they never wait until the regular selling season. Experienced consumers with valuations above x pre-order the product and pay the discounted price $x \leq p$.

Inexperienced consumers do not know their valuations in the advance selling season. An inexperienced consumer has two options. The first is to pre-order and pay x . In this case the consumer's expected payoff is

$$E[v] - x = \mu_0 - x.$$

The other option is to wait until the regular selling season. The consumer learns her valuation v and purchases the product (provided it is in stock) if $v \geq p$. Her expected payoff is

$$(1 - \eta) \int_p^{+\infty} (v - p) f_v(v) dv,$$

where η is the stock-out probability. The stock-out probability is the probability that the consumer will not be able to get the product when she actually wants to purchase it. We use the unconditional density of v , $f_v(v)$, in the integration because the consumer does not know μ .

Thus, inexperienced consumers pre-order if and only if

$$\mu_0 - x \geq (1 - \eta) \int_p^{+\infty} (v - p) f_v(v) dv,$$

or, equivalently,

$$x \leq \hat{x} \equiv \mu_0 - (1 - \eta) \int_p^{+\infty} (v - p) f_v(v) dv.$$

The explicit expression for the threshold value $\hat{x}$ (derived in the Appendix) is

$$\hat{x} = \mu_0 - (1 - \eta) \left((\mu_0 - p) \bar{F}_v(p) + (\sigma^2 + \psi^2) f_v(p) \right), \quad (2)$$

where $\bar{F}_v = 1 - F_v$.

Lemma 1 (Properties of $\hat{x}$). *The threshold value $\hat{x}$, given by (2), has the following properties:*

- (i) $\partial \hat{x} / \partial \sigma < 0$;
- (ii) $\partial \hat{x} / \partial \psi < 0$;
- (iii) $\partial \hat{x} / \partial \eta > 0$.

The properties in this lemma stipulate that inexperienced consumers are less likely to pre-order the more uncertain they are about their valuations (higher σ^2 and/or ψ^2) and the smaller the stock-out probability η is. These properties will be useful for our sensitivity analysis in Section 5.

For the rest of our analysis we will assume $\hat{x} < p$.¹⁴ We consider the following two regions for the advance selling price x .

- Region A: $x \leq \hat{x}$. All inexperienced consumers pre-order.
- Region B: $\hat{x} < x \leq p$. All inexperienced consumers wait until the second period.

4.2 Learning by the Retailer from Pre-orders

When the advance selling price x is in region A, experienced consumers with valuations above x and all inexperienced consumers pre-order. No consumer will wait until the regular selling season. As the retailer's forecast of the second-period demand is $D_2 = 0$, he produces $Q = D_1$.

When x is in region B, the retailer learns μ from observing the first-period demand:

$$D_1 = m_e \text{Prob}(v > x | \mu) = m_e \bar{F}_{v|\mu}(x),$$

that is,

$$D_1 = m_e \left(1 - \Phi \left(\frac{x - \mu}{\sigma} \right) \right),$$

from which μ can be deduced. The retailer produces $Q = D_1 + q$, where q satisfies the second-period demand that is comprised of inexperienced consumers with valuations above p ,

$$D_2 = M_i \text{Prob}(v > p | \mu) = M_i \bar{F}_{v|\mu}(p).$$

Because $M_i \sim \text{LN}(\nu_i, \tau_i^2)$, it is straightforward to show that

$$D_2 \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2).$$

¹⁴Obviously, if $\mu < p$ then $\hat{x} < p$ holds. This sufficient condition essentially means that if all consumers know their valuations, then less than half of them will buy the product at the price p .

4.3 Optimal Value of q

In light of the results of Subsection 4.2, it remains to find the optimal q for

$$D_2 \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2).$$

For any D_2 , if q units are produced then $\min\{q, D_2\}$ units are sold and $(q - D_2)^+ = \max\{q - D_2, 0\}$ are salvaged. The retailer's expected profit from the second period, denoted by π , is

$$\pi(q) = p \mathbb{E}[\min\{q, D_2\}] + s \mathbb{E}[(q - D_2)^+] - cq. \quad (3)$$

The problem of maximizing (3) is the Newsvendor Problem, well-known in the operations management literature. Using the fact that $\min\{q, D_2\} = D_2 - (D_2 - q)^+$, we can rewrite the retailer's expected profit as

$$\pi(q) = \mathbb{E}[D_2](p - c) - \mathbb{E}[(D_2 - q)^+](p - c) - \mathbb{E}[(q - D_2)^+](c - s).$$

The optimal value of q , therefore, minimizes the expected underage and overage costs

$$\mathbb{E}[(D_2 - q)^+](p - c) + \mathbb{E}[(q - D_2)^+](c - s).$$

The first-order condition is

$$\text{Prob}(D_2 \leq q^*) = \beta,$$

where

$$\beta \equiv \frac{p - c}{p - s}.$$

It is clear that q^* selected this way increases in β and therefore increases in the per unit underage cost $p - c$ and decreases in the per unit overage cost $c - s$.

For the lognormal distribution $D_2 \sim \text{LN}(\nu, \tau^2)$ the optimal production quantity is given by

$$q^* = \exp\{\nu + \tau z_\beta\} \quad (4)$$

and

$$\pi^* \equiv \pi(q^*) = (p - s)(1 - \Phi(\tau - z_\beta)) \exp\left\{\nu + \frac{\tau^2}{2}\right\}, \quad (5)$$

where z_β is the β -th percentile of the standard normal distribution, $z_\beta \equiv \Phi^{-1}(\beta)$. See the Appendix for derivations of (4) and (5).

Applying (4) and (5) to $D_2 \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2)$, the optimal q , denoted by $q^*(\mu)$, and the resulting expected profit are

$$q^*(\mu) = \exp\{\nu_i + \tau_i z_\beta\} \bar{F}_{v|\mu}(p) \quad (6)$$

and

$$\pi^*(\mu) \equiv \pi(q^*(\mu)) = (p - s)(1 - \Phi(\tau_i - z_\beta)) m_i \bar{F}_{v|\mu}(p). \quad (7)$$

4.4 Stock-out Probability

In this subsection we determine the equilibrium value of the stock-out probability η .

Given D_2 , the (conditional) probability of any consumer who wants to purchase the product in the regular selling season but is unable to get it is the fraction of excess demand. That is,

$$\text{Prob}(\text{stock-out}|D_2) = \left(\frac{D_2 - q^*}{D_2} \right)^+ = \begin{cases} 0, & D_2 \leq q^*, \\ \frac{D_2 - q^*}{D_2}, & D_2 > q^*. \end{cases}$$

Hence, the stock-out probability is the expected value of this expression over the distribution of D_2 ,

$$\eta = \mathbb{E} \left[\left(\frac{D_2 - q^*}{D_2} \right)^+ \right]. \quad (8)$$

For $D_2 \sim \text{LN}(\nu, \tau^2)$,

$$\eta = \int_{q^*}^{+\infty} \frac{D_2 - q^*}{D_2} g(D_2) dD_2,$$

where $g(D_2)$ is the density function of $\text{LN}(\nu, \tau^2)$ and q^* is given in (4). The explicit expression for the stock-out probability is obtained in the Appendix:

$$\eta = 1 - \beta - \exp \left\{ \tau z_\beta + \frac{\tau^2}{2} \right\} (1 - \Phi(z_\beta + \tau)). \quad (9)$$

Note that this expression is independent of ν . Accordingly, η that results from

$$D_2 \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2)$$

is the same for all μ . This finding is presented in Lemma 2.

Lemma 2 (Stock-out Probability η). *The stock-out probability under the second-period demand $D_2 \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2)$ is the same for all realizations of μ and is given by*

$$\eta = 1 - \beta - \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)).$$

It is worthwhile to note that the stock-out probability in our model is endogenously determined, because q^* is the optimal choice. It follows from Lemma 2 that $\eta < 1 - \beta$, which is expected.

Lemma 3 (Properties of η). *The stock-out probability $\eta = \eta(\beta, \tau_i)$, given in Lemma 2, has the following properties:*

- (i) $\partial \eta / \partial \tau_i > 0$, $\eta(\beta, 0) = 0$, and $\lim_{\tau_i \rightarrow +\infty} \eta(\beta, \tau_i) = 1 - \beta$;
- (ii) $\partial \eta / \partial \beta < 0$, $\eta(0, \tau_i) = 1$, and $\eta(1, \tau_i) = 0$.

Combining the results of Lemma 1(iii) and Lemma 3 we can conclude that the threshold value $\hat{x}$ increases in τ_i and decreases in β .

4.5 The Retailer's Expected Profit

We can now write the retailer's expected total profit Π as a function of the advance selling price x . The part of the retailer's expected profit that comes from experienced consumers equals

$$m_e E_\mu [\text{Prob}(v > x|\mu)] (x - c) = m_e E_\mu [\bar{F}_{v|\mu}(x)] (x - c),$$

as experienced consumers never wait until the regular selling season and only those with valuations above x purchase the product in the advance selling season. In the Appendix we show that

$$E_\mu [\bar{F}_{v|\mu}(x)] = \bar{F}_v(x). \quad (10)$$

The retailer's expected profit from experienced consumers can, therefore, be written as

$$m_e \bar{F}_v(x)(x - c). \quad (11)$$

The purchasing behavior of inexperienced consumers depends on the region that x belongs to. If $x \leq \hat{x}$ (region A), then all inexperienced consumers pre-order. Hence,

$$\Pi^A(x) = m_e \bar{F}_v(x)(x - c) + m_i(x - c). \quad (12)$$

Next, consider $\hat{x} < x \leq p$ (region B). All inexperienced consumers wait until the second period. Since the retailer learns μ , his expected profit from inexperienced consumers is $E_\mu [\pi^*(\mu)]$, where $\pi^*(\mu)$ is given by (7). Hence,

$$\Pi^B(x) = m_e \bar{F}_v(x)(x - c) + E_\mu [\pi^*(\mu)].$$

Let us calculate $E_\mu [\pi^*(\mu)]$:

$$\begin{aligned} \pi^* &\equiv E_\mu [\pi^*(\mu)] = E_\mu [(p - s)(1 - \Phi(\tau_i - z_\beta)) m_i \bar{F}_{v|\mu}(p)] \\ &= (p - s)(1 - \Phi(\tau_i - z_\beta)) m_i E_\mu [\bar{F}_{v|\mu}(p)]. \end{aligned}$$

The term $E_\mu [\bar{F}_{v|\mu}(p)]$ equals $\bar{F}_v(p)$ (see (10)), so we have

$$\pi^* = (p - s)(1 - \Phi(\tau_i - z_\beta)) m_i \bar{F}_v(p). \quad (13)$$

The retailer's expected total profit in region B is, therefore,

$$\begin{aligned} \Pi^B(x) &= m_e \bar{F}_v(x)(x - c) + \pi^* \\ &= m_e \bar{F}_v(x)(x - c) + (p - s)(1 - \Phi(\tau_i - z_\beta)) m_i \bar{F}_v(p). \end{aligned} \quad (14)$$

Combining (12) and (14) yields the retailer's expected total profit as a function of the advance selling price x :

$$\Pi(x) = \begin{cases} \Pi^A(x), & x \leq \hat{x}, \\ \Pi^B(x), & \hat{x} < x \leq p. \end{cases}$$

4.6 Advance Selling vs. no Advance Selling

Before moving onto the optimal advance selling price for the retailer, we explore whether there is always an incentive for the retailer to implement the advance selling strategy.

Without advance selling, the retailer sells in the regular selling season at price p . He chooses Q to maximize

$$E_{\mu} \Pi(Q|\mu),$$

where

$$\begin{aligned} \Pi(Q|\mu) &= p E_{M_i} [\min \{Q, D|\mu\}] + s E[(Q - D|\mu)^+] - cQ, \\ D|\mu &= m_e \bar{F}_{v|\mu}(p) + M_i \bar{F}_{v|\mu}(p). \end{aligned}$$

Note that here Π is a function of output Q , which is a slight abuse of notation. Let Q^0 denote the optimal quantity.¹⁵ The retailer's maximum expected profit without advance selling can be written as

$$\Pi^0 \equiv E_{\mu} \Pi(Q^0|\mu) = \int_{-\infty}^{+\infty} \Pi(Q^0|\mu) f_{\mu}(\mu) d\mu.$$

Below we show that advance selling at price p leads to a higher expected profit than no advance selling, that is, $\Pi(p) > \Pi^0$. The result that the retailer's expected profit under advance selling (at the optimal price which may or may not be equal to p) is greater than that under no advance selling will follow immediately (Proposition 1).

First note that from the retailer's perspective, advance selling at p is equivalent to the no advance selling setting in which the retailer knows the realization of μ . Indeed, in both settings the total demand is $D|\mu = m_e \bar{F}_{v|\mu}(p) + M_i \bar{F}_{v|\mu}(p)$. Only the second term in this expression is stochastic – distributed according to LN $(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2)$. Thus, in both settings the retailer produces

$$m_e \bar{F}_{v|\mu}(p) + q^*(\mu).$$

Let us denote this optimal quantity by $Q^*(\mu)$. Since $Q^*(\mu)$ maximizes $\Pi(Q|\mu)$, $\Pi(Q^*(\mu)|\mu) \geq \Pi(Q^0|\mu)$ with strict inequality for almost all values of μ . It follows that

$$\Pi(p) = \int_{-\infty}^{+\infty} \Pi(Q^*(\mu)|\mu) f_{\mu}(\mu) d\mu > \int_{-\infty}^{+\infty} \Pi(Q^0|\mu) f_{\mu}(\mu) d\mu = \Pi^0.$$

This serves as a proof for the following proposition.¹⁶

Proposition 1 (Advance selling vs. no advance selling). *Advance selling is always superior to no advance selling for the retailer.*

¹⁵This maximization problem does not have an explicit solution.

¹⁶We do not consider the cost of adopting advance selling in this paper. Proposition 1 holds as long as the adoption cost is lower than $\Pi(p) - \Pi^0$.

4.7 Optimal Advance Selling Price

For the rest of our analysis we focus on the case $\hat{x} > c$.¹⁷ Furthermore, we make the following simplifying assumption.

Assumption 1. *The function*

$$\bar{F}_v(x)(x - c)$$

increases in x on $[c, p]$.

This assumption implies that the expected profit from experienced consumers, given by (11), is an increasing function of x for all $x \in [c, p]$.¹⁸ It follows that, as far as experienced consumers are concerned, the retailer has no incentives to offer an advance selling discount. Accordingly, if discounting for pre-orders is offered it must be due to the presence of inexperienced consumers.

It is easy to see that under Assumption 1 the retailer's expected profit $\Pi(x)$ increases in x in each of the two regions A and B. This does not imply, however, that $\Pi(x)$ increases in x on $[c, p]$, as $\Pi(x)$ can jump down at $x = \hat{x}$. A jump down occurs if and only if $\Pi^A(\hat{x}) > \Pi^B(\hat{x})$.¹⁹ That is,

$$m_i(\hat{x} - c) > \pi^* = (p - s)(1 - \Phi(\tau_i - z_\beta))\bar{F}_v(p)$$

(see (12) and (14)). Hence, we have the following two possible patterns for $\Pi(x)$, which are depicted in Figure 2:

- Pattern 1: $\Pi(x)$ jumps down at $\hat{x}$.
- Pattern 2: $\Pi(x)$ jumps up at $\hat{x}$.

Regarding the optimal advance selling price x^* , under pattern 1 it is either $\hat{x}$ or p and under pattern 2 it is p . Advance selling at p corresponds to zero discount for pre-orders. As we have noted in Subsection 4.6, advance selling at p is not equivalent to not offering advance selling, as advance selling at p allows the retailer to learn μ .

Proposition 2 (Optimal advance selling price). *The optimal advance selling price x^* is either $\hat{x}$ or p .*

The two potentially optimal advance selling prices reflect two different tradeoffs for the retailer: between low price-high sales and high price-low sales, and between low price-low expected overage and underage costs and high price-high expected overage and underage costs. Under either price, experienced consumers only buy in the advance selling season. Hence, the tradeoff for the retailer from this group of consumers is only between a lower price and therefore higher expected sales and a higher price and lower expected sales.

For the inexperienced group of consumers, all pre-order at $\hat{x}$; if $x^* = p$, none pre-order and only those with realized values above p will buy in the regular selling season. Hence, both tradeoffs described above are present here. First, $\hat{x}$ corresponds to higher sales and a lower price, while p corresponds to much lower sales and a higher price. Second, $\hat{x}$ means zero overage and underage costs, while p leads to positive expected overage and underage costs.

¹⁷If $\hat{x} < c$, then the optimal advance selling price will be p , as will become clear from Proposition 2 and the discussion preceding it.

¹⁸As it can be shown that $\bar{F}_v(x)(x - c)$ is single-peaked, Assumption 1 holds as long as p is less than or equal to the value at which this function peaks.

¹⁹Since $\Pi_B(x)$ is defined on $(\hat{x}, p]$, $\Pi^B(\hat{x})$ is the limiting value.

Figure 2: The retailer's expected profit as a function of x

5 Sensitivity Analysis

In this section we consider how the retailer's optimal advance selling price x^* and the expected profit $\Pi(x^*)$ studied in the preceding section are affected by some important parameters of the model. Let $\Pi^* \equiv \Pi(x^*)$. Intuitively, we should expect that a decrease in the salvage value s or an increase in the demand uncertainty (captured by ψ and τ_i) would result in a lower Π^* . Surprisingly, we find that Π^* might actually increase (Subsections 5.1 and 5.3).

We are also interested in how the consumer characteristics – the relative number of experienced consumers (measured by parameter α introduced below) and the variation in consumer valuations (captured by σ) – affect the retailer. Will a decrease in α and/or an increase in σ lead to a lower Π^* ? As we show in Subsections 5.2 and 5.3, both a lower and a higher Π^* are possible.

In all of our sensitivity analysis in this section, we focus on small changes in the parameter values so that the retailer's optimal choice stays the same in that it does not jump from $\hat{x}$ to p or vice versa. In Table 3 the directional changes in the parameters (the first row) are chosen to “hurt” the retailer on an intuitive basis. Therefore, all cases in which Π^* increases ($\Pi^* \uparrow$) represent counterintuitive results.

5.1 Sensitivity Analysis – s and τ_i

In this subsection we consider how a decrease in the salvage value s and an increase in the demand uncertainty τ_i affects the retailer.

We first focus on s . Suppose s decreases. Then $\beta = (p - c)/(p - s)$ decreases. By Lemma 3(ii), η increases. An increase in η positively affects $\hat{x}$ (Lemma 1(iii)). As to π^* , it goes down as s decreases. Indeed, applying the Envelope Theorem to (3) yields

$$\left. \frac{\partial \pi}{\partial s} \right|_{q=q^*} = \mathbb{E}[(q^* - D_2)^+] > 0,$$

Figure 3: The effects of a decrease in s (increase in τ_i) on $\Pi(x)$

hence $\pi^*(\mu)$ increases in s , implying $\pi^* = \mathbb{E}_\mu [\pi^*(\mu)]$ increases in s . Geometrically, a decrease in s shifts the threshold $\hat{x}$ to the right and the curve $\Pi^B(x)$ down.

Consider, for example, pattern 1 and suppose $x^* = \hat{x}$. In Figure 3(a) the black and gray curves represent, respectively, $\Pi(x)$ before and after a decrease in s . It is easy to see that both x^* and Π^* increase. If $x^* = p$, then a decrease in s leads to lower Π^* . Consider next pattern 2. The optimal advance selling price is p . As can be seen from Figure 3(b), a decrease in s leads to a lower Π^* . The second column of Table 3 reports the sensitivity results on x^* and Π^* in terms of s .

What is the effect of an increase in τ_i on η , $\hat{x}$, x^* , and Π^* ? By Lemma 3(i), η increases. As a result, $\hat{x}$ increases. Clearly, an increase in τ_i negatively affects π^* (see (13)). It follows that an increase in τ_i affects x^* and Π^* in similar ways as a decrease in s . (See the third column of Table 3.)

The intuition for the above results in regard to a change in the salvage value s is as follows (similar for an increase in τ_i). When s becomes lower, the per unit overage cost becomes higher and, therefore, the retailer reduces his production quantity to avoid too much unsold product at

Table 3: Sensitivity analysis results

parameter optimal choice x^*	change	$s \downarrow$	$\tau_i \uparrow$	$\alpha \downarrow$	σ or $\psi \uparrow$
$\hat{x}$		$x^* \uparrow$ $\Pi^* \uparrow$	$x^* \uparrow$ $\Pi^* \uparrow$	x^* unchanged $\Pi^* \uparrow$	$x^* \downarrow$ $\Pi^* \downarrow$
p		x^* unchanged $\Pi^* \downarrow$	x^* unchanged $\Pi^* \downarrow$	x^* unchanged $\Pi^* \downarrow$	x^* unchanged $\Pi^* \uparrow \downarrow$

the end of the regular selling season. This raises the stock-out probability η for consumers who wait until the regular selling season. As a result, inexperienced consumers are willing to pay a higher advance selling price (i.e., higher $\hat{x}$) to secure delivery of the product. Consider the case in which the retailer's optimal advance selling price $x^* = \hat{x}$. Since the retailer's expected profit from the group of experienced consumers is an increasing function of x (Assumption 1), it becomes higher. Obviously, the retailer's expected profit from the group of inexperienced consumers rises as well since all of them pre-order at a higher price. Hence, the total expected profit of the retailer becomes greater. In this case, we obtain the counterintuitive result that the retailer can benefit from a decrease in s .

The increased stock-out probability does hurt the retailer if the optimal advance selling price $x^* = p$. In this case, the retailer's expected profit from experienced consumers remains the same as there is no change for this group both in price and in the number of buyers. However, the retailer's expected profit from the group of inexperienced consumers, who all wait until the regular selling season, becomes lower due to the increased per unit overage cost. It follows that the retailer is hurt by a decrease in s .

5.2 Sensitivity Analysis – α

Let $m = m_e + m_i$ denote the total expected number of (experienced and inexperienced) consumers, and let α denote the proportion of experienced consumers in the market. Thus, $m_e = \alpha m$ and $m_i = (1 - \alpha)m$. In this subsection we consider how a decrease in α affects the retailer, assuming m is fixed.

Rewriting the expressions (12) and (14) in terms of α yields

$$\Pi^A(x; \alpha) = \alpha m \bar{F}_v(x)(x - c) + (1 - \alpha)m(x - c)$$

and

$$\Pi^B(x; \alpha) = \alpha m \bar{F}_v(x)(x - c) + (p - s)(1 - \Phi(\tau_i - z_\beta))(1 - \alpha)m \bar{F}_v(p).$$

Note that, from Lemma 2 and (2), η and $\hat{x}$ are independent of α . Moreover, the relative order of $\Pi^A(\hat{x}; \alpha)$ and $\Pi^B(p; \alpha)$ is independent of α .²⁰ Hence, x^* stays at the same value as α changes. To see how Π^* is affected, we calculate the derivatives of $\Pi^A(\hat{x}; \alpha)$ and $\Pi^B(p; \alpha)$ with respect to α and show that the first derivative is negative and the second is positive (see the Appendix). Therefore, if $x^* = \hat{x}$ then a decrease in α leads to a higher Π^* , and if $x^* = p$ to a lower Π^* .

The forth column of Table 3 reports the results of our sensitivity analysis with respect to α . The intuition is straightforward. A decrease in α changes the composition of experienced and inexperienced consumers in the total consumer population by decreasing the group size of experienced consumers and increasing the group size of inexperienced consumers. Such a change does not alter the incentive for the retailer to produce to satisfy the demand in the regular selling season and does not affect each individual consumer's incentive to pre-order in the advance selling season. It does, however, affect the retailer's expected profit. In the case in which the retailer's optimal advance selling price $x^* = \hat{x}$, no one buys in the regular selling season. The size of pre-orders placed by experienced consumers decreases while the size of

²⁰Indeed, $\Pi^A(\hat{x}; \alpha) > \Pi^B(p; \alpha)$ if and only if $x - c > (p - s)(1 - \Phi(\tau_i - z_\beta)) \bar{F}_v(p)$, which is independent of α .

pre-orders placed by inexperienced consumers increases. The latter change is greater than the former, because only a fraction of experienced consumers purchase the product. Hence, the retailer's expected profit increases.

If the optimal advance selling price $x^* = p$, the opposite occurs. In this case all experienced consumers whose valuations are above p pre-order but only a fraction of those inexperienced consumers whose valuations are above p get to buy the product in the regular selling season due to the positive stock-out probability. This leads to a net loss in the expected sales for the retailer and therefore to a lower expected profit.

5.3 Sensitivity Analysis – σ and ψ

We will first focus on parameter σ , which captures the variation in consumer valuations. For inexperienced consumers an increase in σ also means they become more uncertain about their valuations.

By Lemma 1(i), $\hat{x}$ decreases as σ increases. In the Appendix we show that whenever $x^* = \hat{x}$, an increase in σ leads to lower expected profit for the retailer. If $x^* = p$, Π^* can increase or decrease. The fifth column of Table 3 reports the results.

The intuition is as follows. When σ increases, inexperienced consumers become less certain about their valuations of the product. As a result, they require better incentives than before to be willing to pre-order in the advance selling season (i.e., lower $\hat{x}$). In the case in which the optimal advance selling price $x^* = \hat{x}$, all consumers who buy pre-order, now at a lower price. By Assumption 1, the retailer's expected profit from experienced consumers becomes lower. The retailer's expected profit from inexperienced consumers falls as well since all of them pre-order at a lower price. Hence, the total expected profit of the retailer becomes smaller.

Consider now the case in which the optimal advance selling price $x^* = p$. As σ increases, the unconditional distribution of consumers' valuations becomes more dispersed in that more consumers have valuations farther away from the mean value than before. As a result, if $p > \mu_0$ then more consumers have valuations above p and if $p < \mu_0$ then less consumers have valuations above p . Suppose $p > \mu_0$. The retailer gets more pre-orders from experienced consumers in the advance selling season and a larger demand from inexperienced consumers in the regular selling season, both yielding greater profit. On the other hand, for $p < \mu_0$ the opposite occurs and the retailer's total expected profit decreases. Thus, in the case $x^* = p$ the directional change in the retailer's expected profit depends on the value of p in relation to μ_0 , both are exogenously given in our model.

Next we consider parameter ψ , which captures the retailer's and consumers' uncertainty about μ . It turns out that ψ has algebraically the same effect on η , $\hat{x}$, x^* , and Π^* as parameter σ . We did not expect this result. Below we provide an explanation for it.

Parameters σ and ψ appear together as the combination $\sigma^2 + \psi^2$ in the unconditional distribution of consumers' valuations function $F_v(v)$. Together with η , $F_v(v)$ matters for inexperienced consumers when they decide whether to pre-order the product (see (2)). The endogenously determined stock-out probability η is not affected by σ nor ψ , since at the point when the retailer chooses q , he only faces uncertainty regarding the number of inexperienced consumers M_i (see Lemma 2). Hence, $\hat{x}$ depends on the combination $\sigma^2 + \psi^2$. The retailer's expected profit

functions

$$\Pi^A(x) = m_e E_\mu [\bar{F}_{v|\mu}(x)] (x - c) + m_i(x - c)$$

and

$$\Pi^B(x) = m_e E_\mu [\bar{F}_{v|\mu}(x)] (x - c) + (p - s) (1 - \Phi(\tau_i - z_\beta)) m_i E_\mu [\bar{F}_{v|\mu}(p)]$$

boil down to

$$\Pi^A(x) = m_e \bar{F}_v(x)(x - c) + m_i(x - c)$$

and

$$\Pi^B(x) = m_e \bar{F}_v(x)(x - c) + (p - s) (1 - \Phi(\tau_i - z_\beta)) m_i \bar{F}_v(p),$$

due to the (purely statistical) fact that $E_\mu [\bar{F}_{v|\mu}(v)] = \bar{F}_v(v)$ when both $v|\mu$ and μ are normally distributed (see Subsection 4.5). It follows that the retailer's profit functions in the two regions depend on the combination $\sigma^2 + \psi^2$.

The result that σ and ψ have identical effects on the equilibrium should be taken with caution, since it depends on the assumptions on the distributions of $v|\mu$ and μ .

6 Advance Selling at a Price Premium

In this section we explore the possibility that the retailer sets an advance selling price above the regular selling price. Let $y > p$ denote such premium price.

Since $\hat{x} < p < y$, inexperienced consumers always wait until the regular selling season. Each experienced consumer compares her payoff from pre-order, $v - y$, with her expected payoff from waiting until the regular selling season, $(1 - \eta)(v - p)$. Later in this section we show that the endogenously determined η depends on μ and y , $\eta = \eta(\mu, y)$.²¹ Experienced consumers observe their valuations, but not μ , so they use $E_{\mu|v}[\eta(\mu, y)]$ in their calculations. In the Appendix we show that the random variable $\mu|v$ is distributed according to

$$\mu|v \sim N\left(\frac{\sigma^2 \mu_0 + \psi^2 v}{\sigma^2 + \psi^2}, \frac{\sigma^2 \psi^2}{\sigma^2 + \psi^2}\right). \quad (15)$$

Let $\hat{v}(y)$ denote the solution to

$$v - y = (1 - E_{\mu|v}[\eta(\mu, y)])(v - p).$$

That is, the experienced consumers with valuations $v \geq \hat{v}(y)$ pre-order the product.

The stock-out probability η is crucial for positive demand at a premium price. In most theoretical papers on advance selling with strategic consumers there is no stock-out risk either because the aggregate market demand is deterministic, or because $s = c$. As a result, the advance selling price should be below the regular selling price to induce some of the consumers to place advance orders. The exceptions are Xie and Shugan (2001), Li and Zhang (2012), and Zhao and Pang (2011). Xie and Shugan (2001) analyze the seller's optimal pricing strategy under limited capacity and show that it is possible to have advance selling prices that are higher

²¹This is contrast to the result in Lemma 2 for the case of advance selling at a discount.

than spot prices. The reason is the same as in our paper: advance buyers must consider both the spot price and the likelihood of unavailable capacity. Li and Zhang (2012) assume uncertain market with heterogeneous consumers. Consumers with the high valuation v_H arrive in the advance selling season, while consumers with the low valuation v_L – in the regular selling season. In this model the regular selling price is equal to v_L and the advance selling price is at a premium. Advance selling discounts do not arise in equilibrium. In Zhao and Pang (2011) consumers also arrive in two different periods: informed consumers – in the advance selling season, uninformed – in the regular selling season. The two types of consumers draw their valuations from the same distribution. Because the model includes the Newsvendor Problem (stochastic market size and inventory risk), the probability of a stock-out is positive, making advance selling at a price premium possible.²²

The retailer learns μ from observing the first-period demand:

$$D_1 = m_e \text{Prob}(v > \hat{v}(y)|\mu) = m_e \bar{F}_{v|\mu}(\hat{v}(y)),$$

that is,

$$D_1 = m_e \left(1 - \Phi \left(\frac{\hat{v}(y) - \mu}{\sigma} \right) \right),$$

from which μ can be obtained. The rest of the experienced consumers with valuations above p (i.e., experienced consumers with $p < v < \hat{v}(y)$) and inexperienced consumers with valuations above p constitute the demand during the regular selling season:

$$D_2 = m_e (\bar{F}_{v|\mu}(p) - \bar{F}_{v|\mu}(\hat{v}(y))) + M_i \bar{F}_{v|\mu}(p).$$

Because the first term of D_2 is non-random and

$$M_i \bar{F}_{v|\mu}(p) \sim \text{LN}(\nu_i + \ln \bar{F}_{v|\mu}(p), \tau_i^2),$$

D_2 is a shifted lognormal distribution. The retailer produces

$$Q = D_1 + m_e (\bar{F}_{v|\mu}(p) - \bar{F}_{v|\mu}(\hat{v}(y))) + q,$$

where q satisfies the random part of D_2 , $M_i \bar{F}_{v|\mu}(p)$. The optimal value of q is given by (6), that is,

$$q^*(\mu) = \exp\{\nu_i + \tau_i z_\beta\} \bar{F}_{v|\mu}(p).$$

Let us now calculate the stock-out probability. If the realization of $M_i \bar{F}(p)$ is greater than q^* , then $M_i \bar{F}(p) - q^*$ of the consumers who constitute D_2 will not get the product. Hence,

$$\eta(\mu, y) = \mathbb{E}_{M_i} \left[\left(\frac{M_i \bar{F}_{v|\mu}(p) - q^*}{D_2} \right)^+ \right].$$

²²Prasad, Stecke, and Zhao (2010) and Zhao and Stecke (2010) restrict the retailer's strategy to advance selling discounts, although in these models price premiums can induce strictly positive demand in the advance selling season.

The system of equations

$$\begin{cases} \eta(\mu, y) = E_{M_i} \left[\left(\frac{M_i \bar{F}_{v|\mu}(p) - \exp\{\nu_i + \tau_i z_\beta\} \bar{F}_{v|\mu}(p)}{m_e(\bar{F}(p) - \bar{F}_{v|\mu}(\hat{v}(y))) + M_i \bar{F}_{v|\mu}(p)} \right)^+ \right] \\ \hat{v}(y) - y = (1 - E_{\mu|\hat{v}(y)}[\eta(\mu, y)]) (\hat{v}(y) - p) \end{cases}$$

implicitly defines $\hat{v}(y)$ and $\eta(\mu, y)$.

Lemma 4 (Expected profit from a premium price). *Advance selling at a premium price y yields the expected profit*

$$\Pi(y) = m_e \bar{F}_v(\hat{v}(y))(y - p) + \Pi(p)$$

to the retailer.

The derivation of $\Pi(y)$ is straightforward. Advance selling at a price premium allows the retailer to learn μ , but uncertainty about the number of inexperienced consumers, M_i , who wait until the regular selling season, remains. The same applies to advance selling at p . Under either y or p the retailer produces the quantity

$$Q = m_e \bar{F}_{v|\mu}(p) + q^*(\mu).$$

The difference between advance selling at y and at p is that, under y , $m_e \bar{F}_{v|\mu}(\hat{v}(y))$ of the total number of transactions, $m_e \bar{F}_{v|\mu}(p) + \min\{q^*(\mu), M_i \bar{F}_{v|\mu}(p)\}$, occur at a premium price, while under p all transactions occur at price p . Hence,

$$\Pi(y) - \Pi(p) = E_\mu [m_e \bar{F}_{v|\mu}(\hat{v}(y))(y - p)] = m_e \bar{F}_v(\hat{v}(y))(y - p).$$

It follows that $\Pi(y) > \Pi(p)$ for any $y > p$, i.e., the retailer is better off selling in advance at a price premium than at p . Observe that $\Pi(y)$ converges to $\Pi(p)$ as y approaches p from above, and it can be shown that $\Pi(y)$ converges to $\Pi(p)$ as $y \rightarrow \infty$. Hence, the optimal advance selling premium price, denoted by y^* , exists.

Recall the two tradeoffs discussed following Proposition 2. In comparing advance selling at y^* and at p , the first tradeoff goes in favor of advance selling at y^* , while the expected overage and underage costs (the second tradeoff) are the same. As a result, advance selling at y^* dominates advance selling at p .

Combining the results of Proposition 2 and Lemma 4, we have the following proposition.

Proposition 3 (Optimal advance selling price when price premium is considered). *When price premium for advance selling is considered, the optimal advance selling price is either $\hat{x}$ or y^* .*

Note that if pattern 1 in Figure 2 prevails, the optimal advance selling price can be either $\hat{x}$ or y^* . If pattern 2 in Figure 2 prevails, the optimal advance selling price is y^* . It follows immediately that $\Pi^A(\hat{x}) > \Pi^B(\hat{x})$ is a necessary condition for the optimal advance selling price to be $\hat{x}$ and that $\Pi^A(\hat{x}) \leq \Pi^B(\hat{x})$ is a sufficient condition for the optimal advance selling price to be y^* .

7 Concluding Remarks

This paper has studied advance selling when the firm faces inexperienced consumers as well as a group of experienced consumers who have prior experience with an earlier version of the product. We find that it is always in the firm's best interest to adopt advance selling. The optimal pre-order price may be at a discount or a premium relative to the regular selling price.

A number of issues are worthy of further investigation. One issue concerns the price in the regular selling season. As in many papers in the literature, we have treated this price as exogenously given. A natural extension of the present research is to allow the retailer to choose the regular selling price. If the retailer can credibly commit to a regular selling price in the advance selling season, then the analyses in the present paper should carry through. Without this commitment, one would need to consider options such as dynamic pricing and pre-order price guarantee (e.g., Zhao and Pang, 2011).

Another issue concerns the assumption that the distribution of valuations is the same for experienced and inexperienced consumers. One straightforward generalization of our model is to assume that the distribution of valuations for experienced consumers is a rightward shift of that for inexperienced consumers (i.e., experienced consumers value the product more than inexperienced consumers on average). We expect many of the results in our paper to continue to hold in this extension. An alternative might be to assume a more general level of correlation between the two distribution functions. Much remains to be explored about this case.

Appendix

Derivation of (1):

We have:

$$\begin{aligned} f_v(v) &= \int_{-\infty}^{+\infty} f_{v|\mu}(v) f_{\mu}(\mu) d\mu \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(v-\mu)^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi\psi^2}} \exp\left\{-\frac{(\mu-\mu_0)^2}{2\psi^2}\right\} d\mu \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi\psi^2}} \exp\left\{-\frac{(\mu-A)^2}{2B^2} - \frac{(v-\mu_0)^2}{2(\sigma^2+\psi^2)}\right\} d\mu, \end{aligned}$$

where

$$A = \frac{\mu_0\sigma^2 + v\psi^2}{\sigma^2 + \psi^2}$$

and

$$B^2 = \frac{\sigma^2\psi^2}{\sigma^2 + \psi^2}.$$

Hence,

$$\begin{aligned}
f_v(v) &= \frac{B}{\sigma\psi\sqrt{2\pi}} \exp\left\{-\frac{(v-\mu_0)^2}{2(\sigma^2+\psi^2)}\right\} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}B^2} \exp\left\{-\frac{(\mu-A)^2}{2B^2}\right\} d\mu \\
&= \frac{B}{\sigma\psi\sqrt{2\pi}} \exp\left\{-\frac{(v-\mu_0)^2}{2(\sigma^2+\psi^2)}\right\} \\
&= \frac{1}{\sqrt{2\pi(\sigma^2+\psi^2)}} \exp\left\{-\frac{(v-\mu_0)^2}{2(\sigma^2+\psi^2)}\right\},
\end{aligned}$$

which immediately implies $v \sim N(\mu_0, \sigma^2 + \psi^2)$.

Derivation of (2):

Below we show that

$$\int_p^{+\infty} (v-p) f_v(v) dv = (\mu_0 - p) \bar{F}_v(p) + (\sigma^2 + \psi^2) f_v(p).$$

The expression (2) will follow immediately. Applying the change of variable $z = (v-\mu_0)/\sqrt{\sigma^2 + \psi^2}$ yields

$$\begin{aligned}
\int_p^{+\infty} (v-p) f_v(v) dv &= \int_{\frac{p-\mu_0}{\sqrt{\sigma^2+\psi^2}}}^{+\infty} (\mu_0 + z\sqrt{\sigma^2+\psi^2} - p) \phi(z) dz \\
&= (\mu_0 - p) \left(1 - \Phi\left(\frac{p-\mu_0}{\sqrt{\sigma^2+\psi^2}}\right)\right) + \int_{\frac{p-\mu_0}{\sqrt{\sigma^2+\psi^2}}}^{+\infty} \frac{z\sqrt{\sigma^2+\psi^2}}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\
&= (\mu_0 - p) \bar{F}_v(p) + \frac{\sqrt{\sigma^2+\psi^2}}{\sqrt{2\pi}} \int_{\frac{p-\mu_0}{\sqrt{\sigma^2+\psi^2}}}^{+\infty} \exp\left\{-\frac{z^2}{2}\right\} d\frac{z^2}{2}.
\end{aligned}$$

Applying the change of variable $u = z^2/2$ yields

$$\begin{aligned}
\int_p^{+\infty} (v-p) f_v(v) dv &= (\mu_0 - p) \bar{F}_v(p) + \frac{\sqrt{\sigma^2+\psi^2}}{\sqrt{2\pi}} \int_{\frac{(p-\mu_0)^2}{2(\sigma^2+\psi^2)}}^{+\infty} \exp\{-u\} du \\
&= (\mu_0 - p) \bar{F}_v(p) + \frac{\sqrt{\sigma^2+\psi^2}}{\sqrt{2\pi}} \exp\left\{-\frac{(p-\mu_0)^2}{2(\sigma^2+\psi^2)}\right\} \\
&= (\mu_0 - p) \bar{F}_v(p) + (\sigma^2 + \psi^2) f_v(p).
\end{aligned}$$

Proof of Lemma 1:

It follows immediately from (2) that $\hat{x}$ increases in η , so we have part (iii) of Lemma 1.

(i) We need to show that $\int_p^{+\infty} (v - p)f_v(v) \, dv$ increases in σ .

$$\begin{aligned}
\int_p^{+\infty} (v - p)f_v(v) \, dv &= \int_p^{+\infty} v f_v(v) \, dv - p \int_p^{+\infty} f_v(v) \, dv \\
&= \int_{-\infty}^{+\infty} v f_v(v) \, dv - \int_{-\infty}^p v \, dF_v(v) - p(1 - F_v(p)) \\
&= \mu_0 - \left(v F_v(v) \Big|_{-\infty}^p - \int_{-\infty}^p F_v(v) \, dv \right) - p + p F_v(p) \\
&= \mu_0 + \int_{-\infty}^p F_v(v) \, dv - p.
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial \sigma} \left(\int_p^{+\infty} (v - p)f_v(v) \, dv \right) = \frac{\partial}{\partial \sigma} \left(\int_{-\infty}^p F_v(v) \, dv \right).$$

Suppose $p < \mu_0$, then

$$\begin{aligned}
\frac{\partial}{\partial \sigma} \left(\int_{-\infty}^p F_v(v) \, dv \right) &= \frac{\partial}{\partial \sigma} \left(\sqrt{\sigma^2 + \psi^2} \int_{-\infty}^{\frac{p - \mu_0}{\sqrt{\sigma^2 + \psi^2}}} \Phi(z) \, dz \right) \\
&= \left(\int_{-\infty}^{\frac{p - \mu_0}{\sqrt{\sigma^2 + \psi^2}}} \Phi(z) \, dz - \frac{p - \mu_0}{\sqrt{\sigma^2 + \psi^2}} \Phi \left(\frac{p - \mu_0}{\sigma} \right) \right) \frac{\sigma}{\sqrt{\sigma^2 + \psi^2}} > 0. \quad (16)
\end{aligned}$$

Next, suppose $p > \mu_0$.

$$\int_{-\infty}^p F_v(v) \, dv = \int_{-\infty}^{2\mu_0 - p} F_v(v) \, dv + \int_{2\mu_0 - p}^{\mu_0} F_v(v) \, dv + \int_{\mu_0}^p F_v(v) \, dv. \quad (17)$$

Applying the change of variable $u = 2\mu_0 - v$ to the second integral yields

$$\int_{\mu_0}^p F_v(2\mu_0 - u) \, du.$$

By symmetry of the normal distribution $F_v(v) + F_v(2\mu_0 - v) = 1$, thus the sum of the last two integrals in (17) equals $p - \mu_0$. Hence,

$$\frac{\partial}{\partial \sigma} \left(\int_{-\infty}^p F_v(v) \, dv \right) = \frac{\partial}{\partial \sigma} \left(\int_{-\infty}^{2\mu_0 - p} F_v(v) \, dv \right).$$

The upper limit of the integration on the right-hand side is less than μ_0 , hence, by (16) the above derivative is positive.

(ii) The proof that $\hat{x}$ decreases in ψ follows the same steps as in (i).

Derivations of (4) and (5):²³

As shown in Silver et al. (1998), the solution to the Newsvendor Problem (3) is q^* that satisfies

$$\Pr(D_2 \leq q^*) = \beta. \quad (18)$$

Moreover,

$$\pi(q^*) = (p - s) \int_0^{q^*} D_2 g(D_2) dD_2.$$

With $D_2 \sim \text{LN}(\nu, \tau^2)$, (18) becomes

$$\Phi\left(\frac{\ln q^* - \nu}{\tau}\right) = \beta,$$

or

$$q^* = \exp\{\nu + \tau z_\beta\}.$$

Next,

$$\pi(q^*) = (p - s) \int_0^{\exp\{\nu + \tau z_\beta\}} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\ln D_2 - \nu)^2}{2\tau^2}\right\} dD_2.$$

Applying the change of variable $u = \ln D_2$ yields

$$\begin{aligned} \pi(q^*) &= (p - s) \int_{-\infty}^{\nu + \tau z_\beta} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u - \nu)^2}{2\tau^2} + u\right\} du \\ &= (p - s) \exp\left\{\frac{(\nu + \tau^2)^2 - \nu^2}{2\tau^2}\right\} \int_{-\infty}^{\nu + \tau z_\beta} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u - (\nu + \tau^2))^2}{2\tau^2}\right\} du \\ &= (p - s) \exp\left\{\nu + \frac{\tau^2}{2}\right\} \Phi(z_\beta - \tau) \\ &= (p - s) (1 - \Phi(\tau - z_\beta)) \exp\left\{\nu + \frac{\tau^2}{2}\right\}. \end{aligned}$$

Derivation of (9):

Let $D_2 \sim \text{LN}(\nu, \tau^2)$ and $q^* = \exp\{\nu + \tau z_\beta\}$. Then

$$\begin{aligned} \eta &= \int_{q^*}^{+\infty} \frac{D_2 - q^*}{D_2} g(D_2) dD_2 \\ &= 1 - G(q^*) - \int_{q^*}^{+\infty} \frac{q^*}{D_2} g(D_2) dD_2 \\ &= 1 - G(\exp\{\nu + \tau z_\beta\}) \\ &\quad - \int_{\exp\{\nu + \tau z_\beta\}}^{+\infty} \frac{\exp\{\nu + \tau z_\beta\}}{D_2} \frac{1}{D_2 \sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\ln D_2 - \nu)^2}{2\tau^2}\right\} dD_2. \end{aligned}$$

²³The solution to the Newsvendor Problem under $\text{LN}(\nu, \tau^2)$ has been provided in lecture notes by Gallego (1995) without a proof.

Applying the change of variable $u = \ln D_2$ yields

$$\begin{aligned}
\eta &= 1 - \beta - \int_{\nu + \tau z_\beta}^{+\infty} \frac{\exp\{\nu + \tau z_\beta\}}{\exp\{u\}} \frac{1}{\exp\{u\} \sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u - \nu)^2}{2\tau^2}\right\} \exp\{u\} \, du \\
&= 1 - \beta - \exp\{\nu + \tau z_\beta\} \int_{\nu + \tau z_\beta}^{+\infty} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u - \nu)^2}{2\tau^2} - u\right\} \, du \\
&= 1 - \beta - \exp\{\nu + \tau z_\beta\} \exp\left\{-\nu + \frac{\tau^2}{2}\right\} \int_{\nu + \tau z_\beta}^{+\infty} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u - (\nu - \tau^2))^2}{2\tau^2}\right\} \, du \\
&= 1 - \beta - \exp\left\{\tau z_\beta + \frac{\tau^2}{2}\right\} \left(1 - \Phi\left(\frac{\nu + \tau z_\beta - (\nu - \tau^2)}{\tau}\right)\right) \\
&= 1 - \beta - \exp\left\{\tau z_\beta + \frac{\tau^2}{2}\right\} (1 - \Phi(z_\beta + \tau)).
\end{aligned}$$

Proof of Lemma 3:

The following properties of the standard normal distribution will be used in the proof:

$$\phi(u) \left(\frac{1}{u} - \frac{1}{u^3}\right) < 1 - \Phi(u) < \frac{\phi(u)}{u}, \quad u > 0 \quad (19)$$

and

$$\phi(u) \left(\frac{1}{|u|} - \frac{1}{|u|^3}\right) < \Phi(u) < \frac{\phi(u)}{|u|}, \quad u < 0. \quad (20)$$

(i) First,

$$\eta(\beta, 0) = 1 - \beta - (1 - \Phi(z_\beta)) = 1 - \beta - (1 - \beta) = 0.$$

Next,

$$\lim_{\tau_i \rightarrow +\infty} \eta(\beta, \tau_i) = 1 - \beta - \lim_{\tau_i \rightarrow +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\} (1 - \Phi(z_\beta + \tau_i)).$$

It follows from (19) that $1 - \Phi(u) = \frac{\phi(u)}{u}(1 + o(u^{-1}))$. Hence, the above expression can be rewritten as

$$\begin{aligned}
\lim_{\tau_i \rightarrow +\infty} \eta(\beta, \tau_i) &= 1 - \beta - \lim_{\tau_i \rightarrow +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\} \frac{\phi(z_\beta + \tau_i)}{z_\beta + \tau_i} \\
&= 1 - \beta - \lim_{\tau_i \rightarrow +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\} \frac{\exp\left\{-\frac{(z_\beta + \tau_i)^2}{2}\right\}}{\sqrt{2\pi}(z_\beta + \tau_i)} \\
&= 1 - \beta - \lim_{\tau_i \rightarrow +\infty} \frac{\exp\left\{-\frac{z_\beta^2}{2}\right\}}{\sqrt{2\pi}(z_\beta + \tau_i)} = 1 - \beta.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\partial \eta}{\partial \tau_i} &= \frac{\partial}{\partial \tau_i} \left(-\exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)) \right) \\
&= -(z_\beta + \tau_i) \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)) + \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \phi(z_\beta + \tau_i) \\
&= (z_\beta + \tau_i) \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \left[\frac{\phi(z_\beta + \tau_i)}{z_\beta + \tau_i} - (1 - \Phi(z_\beta + \tau_i)) \right].
\end{aligned}$$

By (19) the term in the square brackets is positive, so $\partial \eta / \partial \tau_i > 0$.

(ii) First,

$$\eta(0, \tau_i) = 1 - \lim_{z_\beta \rightarrow -\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)).$$

It follows from (20) that $\Phi(u) = \frac{\phi(u)}{|u|} (1 + o(u^{-1}))$. Hence, the above expression can be rewritten as

$$\begin{aligned}
\eta(0, \tau_i) &= 1 - \lim_{z_\beta \rightarrow -\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \left(1 - \frac{\phi(z_\beta + \tau_i)}{|z_\beta + \tau_i|} \right) \\
&= 1 - \lim_{z_\beta \rightarrow -\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \left(1 - \frac{\exp \left\{ -\frac{(z_\beta + \tau_i)^2}{2} \right\}}{\sqrt{2\pi} |z_\beta + \tau_i|} \right) \\
&= 1 - \lim_{z_\beta \rightarrow -\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} + \lim_{z_\beta \rightarrow -\infty} \frac{\exp \left\{ -\frac{z_\beta^2}{2} \right\}}{\sqrt{2\pi} |z_\beta + \tau_i|} = 1.
\end{aligned}$$

Next,

$$\eta(1, \tau_i) = - \lim_{z_\beta \rightarrow +\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)).$$

Since $1 - \Phi(u) = \frac{\phi(u)}{u} (1 + o(u^{-1}))$,

$$\begin{aligned}
\eta(1, \tau_i) &= - \lim_{z_\beta \rightarrow +\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \frac{\phi(z_\beta + \tau_i)}{z_\beta + \tau_i} \\
&= - \lim_{z_\beta \rightarrow +\infty} \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \frac{\exp \left\{ -\frac{(z_\beta + \tau_i)^2}{2} \right\}}{\sqrt{2\pi} (z_\beta + \tau_i)} \\
&= - \lim_{z_\beta \rightarrow +\infty} \frac{\exp \left\{ -\frac{z_\beta^2}{2} \right\}}{\sqrt{2\pi} (z_\beta + \tau_i)} = 0.
\end{aligned}$$

Finally, we can write η as a function of z_β and τ_i ,

$$\eta(z_\beta, \tau_i) = 1 - \Phi(z_\beta) - \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)).$$

Differentiating the above expression with respect to z_β yields

$$\frac{\partial \eta}{\partial z_\beta} = -\phi(z_\beta) - \tau_i \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)) + \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \phi(z_\beta + \tau_i).$$

The first term and the third term cancel out each other, as

$$\exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} \frac{\exp \left\{ -\frac{(z_\beta + \tau_i)^2}{2} \right\}}{\sqrt{2\pi}} = \frac{\exp \left\{ -\frac{z_\beta^2}{2} \right\}}{\sqrt{2\pi}}.$$

Thus,

$$\frac{\partial \eta}{\partial z_\beta} = -\tau_i \exp \left\{ \tau_i z_\beta + \frac{\tau_i^2}{2} \right\} (1 - \Phi(z_\beta + \tau_i)) < 0.$$

Since z_β is increasing in β , it follows immediately that $\partial \eta / \partial \beta$ is also negative.

Derivation of (10):

We have:

$$\begin{aligned} E_\mu [\bar{F}_{v|\mu}(x)] &= \int_{-\infty}^{+\infty} \bar{F}_{v|\mu}(x) f_\mu(\mu) \, d\mu = \int_{-\infty}^{+\infty} \left(\int_x^{+\infty} f_{v|\mu}(v) \, dv \right) f_\mu(\mu) \, d\mu \\ &= \int_{-\infty}^{+\infty} \int_x^{+\infty} f_{v|\mu}(v) f_\mu(\mu) \, dv \, d\mu = \int_x^{+\infty} \int_{-\infty}^{+\infty} f_{v|\mu}(v) f_\mu(\mu) \, d\mu \, dv. \end{aligned}$$

The integral

$$\int_{-\infty}^{+\infty} f_{v|\mu}(v) f_\mu(\mu) \, d\mu$$

is $f_v(v)$, hence

$$E_\mu [\bar{F}_{v|\mu}(x)] = \int_x^{+\infty} f_v(v) \, dv = \bar{F}_v(x).$$

Derivatives of $\Pi^A(\hat{x})$ and $\Pi^B(p)$ with respect to α :

Differentiating $\Pi^A(\hat{x}; \alpha)$ with respect to α yields

$$\frac{\partial \Pi^A(\hat{x}; \alpha)}{\partial \alpha} = m \bar{F}_v(\hat{x})(\hat{x} - c) - m(\hat{x} - c) < 0.$$

Differentiating $\Pi^B(p; \alpha)$ with respect to α yields

$$\begin{aligned} \frac{\partial \Pi^B(p; \alpha)}{\partial \alpha} &= m \bar{F}_v(p)((p - c) - (p - s)(1 - \Phi(\tau_i - z_\beta))) \\ &= m \bar{F}_v(p)((p - s)\Phi(\tau_i - z_\beta) - (c - s)) \\ &> m \bar{F}_v(p)((p - s)\Phi(-z_\beta) - (c - s)). \end{aligned}$$

We substitute $\Phi(-z_\beta) = 1 - \Phi(z_\beta) = 1 - \beta = (c - s)/(p - s)$ into the above inequality:

$$\frac{\partial \Pi^B(p; \alpha)}{\partial \alpha} > m \bar{F}_v(p)((c - s) - (c - s)) = 0.$$

Derivatives of $\Pi^A(\hat{x})$ and $\Pi^B(p)$ with respect to σ :

The expected profit function in region A, given by (12), depends on σ :

$$\begin{aligned}\Pi^A(x; \sigma) &= m_e \bar{F}_v(x)(x - c) + m_i(x - c) \\ &= m_e \left(1 - \Phi \left(\frac{x - \mu_0}{\sqrt{\sigma^2 + \psi^2}} \right) \right) (x - c) + m_i(x - c).\end{aligned}$$

For any x in region A, $x < \mu_0$, so

$$\frac{\partial}{\partial \sigma} \left(1 - \Phi \left(\frac{x - \mu_0}{\sqrt{\sigma^2 + \psi^2}} \right) \right) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left\{ -\frac{(x - \mu_0)^2}{2(\sigma^2 + \psi^2)} \right\} \frac{\sigma(x - \mu_0)}{(\sigma^2 + \psi^2)\sqrt{\sigma^2 + \psi^2}} < 0.$$

It follows that

$$\frac{d\Pi^A(\hat{x}; \sigma)}{d\sigma} = \frac{\partial \Pi^A(\hat{x}; \sigma)}{\partial \sigma} + \frac{\partial \Pi^A(\hat{x}; \sigma)}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial \sigma} < 0,$$

because the first term is negative as was shown above, and the second term is negative because $\partial \Pi^A(\hat{x}; \sigma)/\partial \hat{x} > 0$ by Assumption 1 and $\partial \hat{x}/\partial \sigma < 0$ by Lemma 1(ii). Next, we determine the sign of $\partial \Pi^B(p; \sigma)/\partial \sigma$. As can be seen from (14), the sign of this derivative equals the sign of

$$\frac{\partial}{\partial \sigma} \left(1 - \Phi \left(\frac{p - \mu_0}{\sqrt{\sigma^2 + \psi^2}} \right) \right),$$

which in turn equals the sign of $p - \mu_0$, which could be positive or negative.

Derivation of (15):

We have:

$$\begin{aligned}f_{\mu|v}(\mu) &= \frac{f_{v|\mu}(v)f_{\mu}(\mu)}{f_v(v)} \\ &= \frac{1}{2\sqrt{\frac{\pi\sigma^2\psi^2}{\sigma^2+\psi^2}}} \exp \left\{ -\frac{(v - \mu)^2}{2\sigma^2} - \frac{(\mu - \mu_0)^2}{2\psi^2} + \frac{(v - \mu_0)^2}{2(\sigma^2 + \psi^2)} \right\}.\end{aligned}$$

Regrouping the terms in the exponent yields

$$f_{\mu|v}(\mu) = \frac{1}{2\sqrt{\frac{\pi\sigma^2\psi^2}{\sigma^2+\psi^2}}} \exp \left\{ -\frac{\left(\mu - \frac{\sigma^2\mu_0 + \psi^2 v}{\sigma^2 + \psi^2} \right)^2}{\frac{2\sigma^2\psi^2}{\sigma^2 + \psi^2}} \right\},$$

which immediately implies (15).

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