Rescuing the Financial System: Capabilities, Incentives, and Optimal Interbank Networks

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Abstract: I model bank rescues in a setting where banks hold each other’s financial instruments, creating a network of financial linkages. Costly bankruptcies reduce interbank payments, which creates incentives for rescues by other banks. I show that the government’s bail-out costs are minimized if regulators promote financial networks that are evenly connected and have intermediate levels of interbank liabilities but interestingly have low diversification at the bank level. Such networks maximize banks’ contributions to the rescue of a distressed bank hit by a relatively small negative shock, but also ensure that banks do not fail sequentially like dominos when a bank hit by a large shock does actually fail. The results also provide a rationale for why some systemically important banks such as Lehman Brothers or Washington Mutual were not rescued in 2007-2008. In the model, a welfare-maximizing government assists the rescues designed to prevent domino failures and maintain financial stability instead of assisting the rescue of a bank that is hit by a large shock.

Keywords: bail-in, bail-out, bank rescues, contagion, financial networks, financial stability, rescue mergers, systemic risk.

JEL Classification Codes: D85, G21, G28, G34, H81.
The global financial system is connected through the interbank market that helps financial institutions to cope with individual failure risks. However, failures, if they occur, might snowball into a systemic crisis due to the contagion of counterparty risk. The default of a bank on its interbank debt may trigger the default of its counterparties, and so on. Although the financial systems evolve over time, there is a historical debate over bank rescues and financial stability, going back to Thornton (1802) and Bagehot (1873). Bail-out is the resolution method in which the government, as being the lender of last resort, provides the financial support to the banks that face bankruptcy risk. However, bail-outs might be too costly to the government. Accordingly, the costly bail-outs during the 2007-2008 global financial crisis have brought private-sector resolution methods, in which distressed banks are rescued by other banks, to the forefront of the discussions. The private-sector resolution of a distressed bank might be designed with or without government assistance, and might be in the form of a merger of two banks or a bail-in by a consortium of multiple banks. The advantage of using a private-sector resolution method in any form, which I call interbank rescue, is that it reduces the government’s role in bank rescues and, thus, the burden on the taxpayers. However, it is costly for the rescuer banks. Then, a fundamental question arises: When do banks have incentives and capabilities to rescue the distressed banks internally? In this paper, I study the effectiveness of interbank rescues, which might be thought of as a self-correcting mechanism of the financial system that can be used when a set of banks face bankruptcy risks.

1"If any one bank fails, a general run on the neighboring ones is apt to take place, which if not checked at the beginning by a pouring into the circulation a large quantity of gold, leads to very extensive mischief.” (Thornton, 1802, p. 113)

2“A panic, in a word, is a species of neuralgia, and according to the rules of science you must not starve it. The holders of the cash reserve must be ready not only to keep it for their own liabilities, but to advance it most freely for the liabilities of others. They must lend to merchants, to minor bankers, to “this man and that man”, wherever the security is good. In wild periods of alarm, one failure makes many, and the best way to prevent the derivative failures is to arrest the primary failure which caused them.” (Bagehot 1873, pp. 51–2)

Hoggarth, Reidhill and Sinclair (2004), and White and Yorulmazer (2014) discuss bank resolution concepts and practices in detail.
Besides the extensive literature on the financial contagion, there have been a few theoretical studies focusing on banks’ rescue actions in financial networks. The objective of this paper is twofold. The first objective is to investigate the role of the architecture of the financial network in banks’ rescue capabilities and incentives. Accordingly, I analyze the sources of inefficiencies in interbank rescues, and investigate the welfare effects of the government assistance in bank rescues. The second objective is to provide optimal interbank network structures, in which the internal rescue mechanism works most effectively against failure risks and minimizes the government’s role in bank rescues.

The resolution of Long-Term Capital Management (LTCM) in 1998 is an example of a non-assisted multi-bank bail-in.\(^3\) Report of the President’s Working Group on Financial Markets explains the bail-in of LTCM by its counterparties as follows:

“The firms in the consortium saw that their losses could be serious, with potential losses to some firms amounting to $300 million to $500 million each.... The self-interest of these firms was to find an alternative resolution that cost less than they could expect to lose in the event of default.”

(Rubin et al., 2009)

In LTCM case, the potential losses being greater than the rescue costs had triggered the rescue. The story was quite different in the 2007-2008 financial crisis. The Wall Street rescue consortium had been organized by the Federal Reserve Bank of New York to help rescue Lehman Brothers similar to the case in LTCM. Meanwhile, Lehman Brothers negotiated with various organizations for a potential merger. Geithner (2015) explains the bail-in and rescue merger attempts in Lehman Brothers case as follows:

“We told the bankers from the night before to divide themselves into three groups: one to analyze Lehman’s toxic assets to help facilitate

\(^3\)The Federal Reserve Bank of New York organized a consortium to save LTCM. The 14-member consortium injected around $3.6 billion dollars and collectively received a 90 percent equity stake in LTCM and LTCM shareholders received the remaining 10 percent, worth about $400 million (Rubin et al., 2009).
a potential merger, one to investigate an LTCM-style consortium that could take over the firm and gradually wind down its positions, and one to explore ways to prepare for a bankruptcy and limit the attendant damage.” (Geithner, 2015)

Consequently, neither the rescue by a consortium or the government nor a merger had been realized, and Lehman Brothers filed for bankruptcy. In the aftermath, the Fed organized a Wall Street consortium to save AIG. However, the consortium members decided not to go through with the rescue, and the Fed assisted AIG in meeting its obligations. In the meantime, most of the largest institutions had been involved in government-assisted rescue mergers to maintain financial stability. In light of the experience of the 2007-2008 financial crisis, an additional set of questions arises: How can we explain the selective decisions of banks and the government to rescue or not to rescue some of the systemically important institutions? Can these decisions be ex- post optimal both for the banks and the government? This study provides plausible answers to these questions, and offers explanations for the selective rescue actions in 2007-2008.

In this paper, I develop a theoretical model of bank rescues, which is built on the contagion model introduced by Elliott, Golub, and Jackson (2014). The financial network is defined over exogenously given interbank contracts. A default occurs when a bank’s total assets are lower than its total liabilities. A bank that defaults liquidates its assets, which incurs bankruptcy costs to the system. I consider that a single bank in the financial network is hit by the shock. The failure of the shocked bank, if it is not stopped, might initiate a cascade of failures by contagion through the interbank linkages. Rescue consortia are formed after the shock hits but before failures occur in the financial network. The rescue consortia game is a simultaneous move game, and the

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4The mega-mergers in that period are as follows (the acquiring institution(s) - the acquired institution (the date)): RBS, Fortis, and Banco Santander - ABN Amro (October 2007); JPMorgan Chase - Bear Stearns (March 2008); Banco Santander - Alliance&Leicester (July 2008); Bank of America - Merrill Lynch (September 2008); Lloyds - HBOS (September 2008 to January 2009), Wells Fargo - Wachovia (October 2008), BNP Paribas - Fortis (May 2009).
solution concept is coalition-proof Nash equilibrium. In this environment, the potential losses due to the interbank linkages create incentives for the formation of rescue consortia. Therefore, the architecture of the interbank network plays a key role in rescue decisions. The integration and the diversification are the two main specifications of the interbank dependencies. The integration level of a bank is the ratio of its total interbank liabilities to its total assets. On the other hand, the diversification level summarizes the information on how many creditors each bank has.

In addition to the network structure, the magnitude of the initial shock plays a key role in rescue capabilities. The rescue of the shocked bank without government assistance becomes infeasible when the shock is higher than a threshold level, since the available capital in the system to rescue the shocked bank becomes inadequate in that case. Then, one might naturally think that the government assistance is essential in order to prevent an imminent cascade of failures. However, the results show the banks can use an alternative resolution method to maintain financial stability: stopping the domino failures that would be triggered by the failure of the shocked bank. Then, a fundamental question arises: Is this alternative method always feasible? I show that rescuing the rest of the system is feasible as long as the integration is sufficiently low. The intuition behind this result is as follows. In highly integrated networks, the failure of the shocked bank incurs high amounts of losses to the rest of the system that makes the alternative method infeasible without government assistance. Following this result, this paper adds a novel insight into the degree of connectivity: the “robust-yet-unpreventably fragile” nature of financial networks. This feature follows from the “robust-yet-fragile” nature of financial networks and can be explained as follows. The imminence of a cascade of failures in a highly diversified network implies that the integration level

5The “robust-yet-fragile” nature of financial networks has been highlighted in many studies in the financial contagion literature. Acemoglu et al. (2015), Cabrales et al. (2017), Elliot et al. (2014), Gai and Kapada (2010) and Nier et al. (2007) show the “robust-yet-fragile” nature of financial networks, which explains the fact that increase in the number of connections increase the robustness of the system, yet it becomes fragile since many banks fail together if a problem occurs in these networks.
is high. Otherwise, the losses after the initial failure would be small enough and be safely absorbed by the other banks. Therefore, in a highly diversified network in which a cascade of failures is imminent, the initial failure induces high amount of losses to the rest of the system, which result in unstoppable contagious failures. From another perspective, when a cascade of failures is imminent in a highly diversified network, many banks would fail together following the initial failure and, thus, the number of potential rescuers becomes too low. Consequently, an increase in the number of connections increases the robustness of the system, but the financial system becomes unpreventably fragile.

Following the findings on the rescue capabilities, next, I focus on rescue incentives. The main insight into the rescue incentives is that the banks which are least affected by the initial failure have the least incentives to prevent that failure. In this sense, the connectedness gains importance for increasing the contributions in the rescue of the initially distressed banks. The reason is that if there exist banks that have no interbank linkages to the contagious part of the network, which implies the disconnectedness of the network, then these banks neither have losses from failures nor gains from costly rescues. Then, the lack of rescue incentives for some banks limit the available capital for rescues, and the government assistance becomes essential to prevent any local contagion. Similar to the disconnectedness case, weak interbank ties in connected networks also cause inefficiencies in bank rescues. More specifically, weak interbank ties result in selective rescue actions, which can be explained as follows. For large shocks, rescuing the shocked bank becomes too costly. In that case, if the contagious failures are imminent but preventable, then the banks prefer to use the less costly resolution method: stopping the domino failures instead of rescuing the shocked bank. On the other hand, if there exist no potential contagious failures, which implies that the dependencies to the shocked bank are weak, then, similar to the previous case, the banks prefer not to rescue the shocked bank, since the rescue costs become larger than the potential gains from the rescue. These two scenarios together show that the banks decide not to rescue a bank that is hit by a large shock whenever the
ratio of “shock to potential losses” is sufficiently high.

The inefficiencies in rescues, which are explained above, can be eliminated via increasing the interbank dependencies, so the interbank liabilities per bank. The intuition here is as follows. Higher the interbank liabilities per bank, higher the potential losses due to any failure and hence more the rescue incentives. However, the results so far show that if the interbank liabilities are sufficiently high in high diversified networks, then the financial system becomes unpreventably fragile. Correspondingly, I provide results on optimal interbank linkages while considering the trade-off that the interbank liabilities entail. I define the optimization problem for the interbank linkages under the constraint of homogeneity in bank sizes and with no constraint on the network architecture. The homogeneity in bank sizes ensures that the potential inefficiencies in rescues arise due to the network structure. Accordingly, I show that a class of optimal networks that minimize the realized bankruptcy costs is evenly connected in which each bank borrows/lends at intermediate levels (intermediate integration) from/to a small number of banks (low diversification). In such networks, a system-wide contagion risk emerges, which incentivizes all banks to stop the initial failure as long as it is feasible. Moreover, sufficiently low integration ensures that banks are always able to rescue the rest of the system without government assistance whenever rescuing a bank hit by a large shock is infeasible. As a result, such optimal networks minimize the government’s role of being the lender of last resort in bank rescues. The result in this part emphasizes the importance of the design of the interbank linkages. In addition to the bank level decisions of diversification and the integration, the system-wide design of the interbank network plays a key role in the extent of contributions in rescue. Evenly connected networks, which are accompanied by intermediate integration and low diversification per bank, are preferred to clustered/disconnected networks since banks’ contributions in rescue of an initially distressed bank is maximized in such evenly connected

6One might worry about the core-periphery structure of the financial networks. In this regard, the main focus of this study is the core banks in a core-periphery environment, in which the rescue of an initially distressed core bank becomes systemically important.
Lastly, I introduce the government assistance in bank rescues. The results in this part show that, for large shocks, the government prefers to assist in the rescue of the rest of the system to stop the domino effects to maintain financial stability, instead of assisting in the rescue of the shocked bank. Therefore, the government-support programs might exclude banks that are hit by large shocks since the rescue of these banks become relatively costly. The results together offer plausible explanations why some of the core banks (i.e., Lehman Brothers, Washington Mutual) had not been saved, but many mega-mergers among the remaining core banks had been formed with the government-assistance during the 2007-2008 financial crisis.

Related literature.—Financial networks attracted considerable attention in the literature after the early contributions by Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000). The implications of different network structures on contagion have been studied in a variety of papers. Gai and Kapada (2010) and Nier et al. (2007) discuss the trade-offs in the degree of connectivity. Elliott, Golub, and Jackson (2014) introduce a contagion model and show that both the diversification and the integration entail trade-offs in how they affect contagion. They show that the financial contagion more likely occurs when there are intermediate levels of integration and diversification in the financial network. Cabrales, Gottardi and Vega-Redondo (2017) study the optimal connectivity for different shock distributions. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015a) show that the types of networks that are most prone to contagious failures depend on the magnitude and the number of shocks affecting the financial network. Glasserman and Young (2014) derive bounds on the potential magnitude of the network effects on contagion while considering various bank characteristics. Cabrales, Gale, and Gottardi (2015) provide the detailed comparison of the last four papers mentioned above. Allen and Babus (2008) discuss the models of networks in finance.

\(^7\)In addition to these, Allen, Babus, and Carletti (2012), Amini and Minca (2016), Battiston (2012), Gofman (2017), and Minca and Sulem (2014) are other relevant papers.
are several papers focusing on the endogenous network formation.\textsuperscript{8} Besides the extensive literature on contagion in financial networks, the theoretical papers on bank rescues in financial networks are limited. Erol (2016) builds a model of network formation and studies the network hazards and bail-outs. Acharya and Yorulmazer (2007) focus on bank resolution and correlation in investment decisions of banks. Leitner (2005) focuses on the ex-ante optimal size of completely connected components in a bail-in model. Rogers and Veraart (2013) show that the incentives for bank rescues arise in the presence of bankruptcy costs and provide results for canonical network structures.

Lastly, Bernard, Capponi, and Stiglitz (2017), in a simultaneous and independent work to mine, study the government bail-outs and subsidized bail-ins.\textsuperscript{9} Bernard et al. (2017) show that the sparsely connected network is socially preferred to the densely connected network by showing that banks’ contribution to a coordinated bail-in are larger in the ring network than in the complete network.\textsuperscript{10} This study differs from Bernard et al. (2017) in two ways. First, I highlight a novel channel in bank rescues: for large shocks, bail-ins (with or without government-assistance) might be designed to stop the domino failures instead of rescuing the failure of the shocked bank. The results with this specification of the model provide plausible explanations for the rescue practices during the 2007-2008 financial crisis. Second, different from Bernard et al. (2017), I analyze the optimal interbank networks in which the interbank rescue mechanism works most effectively and minimizes the government’s role in bank rescues. Accordingly, I incorporate the importance of the connectivity into the analysis. When the complexity of the interbank network in real-life


\textsuperscript{9}Bernard et al. (2017) and Kanik (2017) were presented at the Third Annual NSF Conference on Network Science in Economics at Washington University in St. Louis in April, 2017.

\textsuperscript{10}Bernard et al. (2017) use a similar interbank network model to Acemoglu et al. (2015a) and Rogers and Veraart (2013), and use Eisenberg and Noe (2001) algorithm for clearing of payments. Differently, I use the cross-holdings model introduced by Elliott et al. (2014) to model the interbank network and use their algorithm for clearing of payments. Differences in modeling the interbank network do not alter the results.
is considered, sparse connections might potentially cause clustering or disconnectedness in the financial network, which might cause inefficiencies in bank rescues. This paper shows that the sparse connections should be accompanied by a well-designed network structure in which banks are evenly connected. As a result, together with the density and the strength of the interbank linkages, this paper brings the issue of the system level design of the interbank network into the discussions.

The remainder of the paper is organized as follows. In Section II, I introduce the financial contagion model and the formation of rescue consortia. Section III and IV include the analyses of the rescue capabilities and incentives. Section V discusses the optimal interbank networks, and Section VI discusses the government-assistance in rescues. Section VII concludes.

II. Model

I built the bank rescue framework on the financial contagion model introduced by Elliot, Golub, and Jackson (2014). There is a set \( N = \{1, ..., n\} \) of banks. Each bank \( i \) holds an exogenously given proprietary asset that yields return \( p_i \in \mathbb{R}^N_+ \). The proprietary asset of a bank might be thought of as its own project generating cash flow (e.g., interest-earning loans). In addition, each bank \( i \) has exogenously given external liabilities \( l_i \). These external liabilities exclude the interbank obligations and might be thought of as operational expenses (e.g., wage or tax obligations) or obligations to non-bank creditors. In addition to the proprietary asset and the external liabilities of each bank, the banks have obligations to each other via debt contracts corresponding to the interbank liabilities. The debt contracts among the banks are represented as cross-holdings, which are given exogenously. The cross-holding \( C_{ij} \geq 0 \) that bank \( i \) holds in bank \( j \) implies that bank \( i \) is a creditor of bank \( j \), and claims \( C_{ij} \) portion of the total assets of bank \( j \). The financial network with the given cross-holdings can be represented as a weighted directed graph, where the \( C \) matrix is the cross-holdings matrix, in which \( C_{ii} = 0 \) and \( \sum_{j \in N} C_{ji} < 1 \) for all \( i \). The total assets of bank \( i \) is denoted by \( V_i \), and equal to the sum of its
proprietary asset return, $p_i$, and the interbank assets that it claims in other banks. The total assets of bank $i$ can be written as follows:

$$V_i = \left( \sum_{j \in N} C_{ij} V_j \right) + p_i$$

(1)

In matrix notation:

$$V = CV + p$$

$$V = (I - C)^{-1} p$$

(2)

The total liabilities of bank $i$ is denoted by $L_i$, and it is equal to the sum of its external liabilities and interbank liabilities, which is given by:

$$L_i = \left( \sum_{j \in N} C_{ji} V_i \right) + l_i$$

(3)

Lastly, the shareholders’ equity (or the net worth) of bank $i$ is equal to:

$$e_i = \max \{0, V_i - L_i\}$$

(4)

For simplicity, I consider that there exist no cross-ownership among banks. The net worth of bank $i$ is greater than zero whenever its total assets are greater than its total liabilities. A negative value for the net worth is not allowed since the default occurs in that case, which is explained in detail in the next part. Following the equations (1) to (4), the shareholders’ equity can

\[\text{This implies that there exists no conflict of interest among the shareholders of a given bank. Incorporating the cross-ownerships into the rescue model is a future research direction.}\]
be rewritten as:

\[ e_i = \max\{0, V_i - \left(\sum_{j \in N} C_{ji} V_i\right) - l_i\} = \max\{0, \hat{C}_{ii} V_i - l_i\} \tag{5} \]

where \( \hat{C}_{ii} := 1 - \sum_{j \in N} C_{ji} > 0 \) is the portion of the total assets of bank \( i \) which is not claimed by the other banks in the network.\(^{12}\)

Equation (1) and (2) capture the interdependencies in total assets among banks via the interbank holdings. Equation (3) shows the balance sheet implications of changes in asset values and implies that when the total assets of bank \( i \) (\( V_i \)) decreases by one unit, its interbank liabilities decrease by \( \sum_{j \in N} C_{ji} < 1 \) units.

This shows the linearities in interbank debt contracts in the model.\(^{13}\) The linearities in the interbank debt contracts might be thought of as the voluntarily write-downs in distressed environments. Given these specifications, a financial network is represented by \((C, F)\) where \( C \) is the cross-holdings matrix representing the network characteristics, and \( F \) is the bank characteristics including the information on the proprietary asset returns and the external liabilities of each bank in the network. In the next part, I define the shock environment and discuss how the financial shock might result in the first failure and subsequently the contagious failures.

### A. The Shock, the First Failure and the Contagion

An exogenously given negative shock at an amount of \( s \in [0, p_r] \) hits the proprietary asset of a single bank, called bank \( r \).\(^{14}\) As an example, one can consider

\(^{12}\) \( \hat{C}_{ii} \) is assumed to be strictly positive. In matrix notation, \( \hat{C} \) is an \( n \times n \) diagonal matrix such that \( \hat{C}_{ii} > 0 \ \forall i \) and \( \hat{C}_{ij} = 0 \ \forall i \neq j \). By this assumption, the inverse \( (I - C)^{-1} \) is well defined and non-negative.

\(^{13}\) The linearities in interbank contracts in this model is different from the standard interbank contract models, yet provides a tractable model of contagion in distressed environments and do not alter the main results. Figure 5 in the Appendix depicts the interdependencies in balance sheets.

\(^{14}\) The single shock environment is enough to reveal the idea of the financial contagion and the rescue analysis.
a large drop in a single bank’s returns on individual loans (e.g., a decrease in the mortgage loan repayments by households). The vector of proprietary asset returns after the shock is equal to \( p = [p_1, ..., p_r - s, ..., p_n]^\prime \). The payments to the external and internal creditors are realized simultaneously after the shock hits. The bankruptcy condition is described as follows. If there exists a bank \( i \) such that \( V_i < L_i \), then bank \( i \) enters into a bankruptcy procedure and liquidates its proprietary asset. Liquidation is costly, and \( p_i \) drops by \( \beta_i \) after the liquidation. Thus, \( \beta_i \) might be thought of as the bankruptcy cost of bank \( i \) on its proprietary asset. If bank \( i \) defaults, the shareholders of bank \( i \) receives nothing, and both the external and the internal creditors of the failed bank are rationed in proportion to \( V_i \) with equal seniority. Hence, bank \( j \) claims \( C_{ji} V_i \). The equal seniority assumption is a simplifying assumption which does not alter the main results and the insights into the contagion model.

Following equation (5), the shareholders’ equity is given by \( \max \{0, v_i - l_i\} \) where \( v_i = \hat{C}_{ii} V_i \), and bank \( i \) defaults if \( v_i < l_i \). By including the bankruptcy costs, \( v \) can be written in matrix notation as follows:

\[
\mathbf{v} = \hat{C}(I - C)^{-1}(\mathbf{p} - \mathbf{b}) = A(\mathbf{p} - \mathbf{b})
\]

where \( \mathbf{b} \) is the vector of bankruptcy costs, and \( b_i = \beta_i \) if bank \( i \) defaults, and 0 otherwise. The interbank payments and the shareholders’ equity of each bank are determined simultaneously via the payment solution satisfying equation (6). There always exists a payment solution and there can be multiple solutions. As in Elliott et al. (2014), I focus on the best-case solution in which as few banks as possible fail.

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15 For \( s = 0 \), all banks are always able to pay back their external and interbank liabilities in full and the net worth of each bank is non-negative.

16 \( A = \hat{C}(I - C)^{-1} \) matrix is a column stochastic matrix, called the dependency matrix, which is reminiscent of Leontief (1951), and useful for tracking the contagion in a given financial network.

17 There exists two main sources of the multiplicity of the payment solution. One source of the multiple solution is based on the story of self-fulfilling failures similar to the bank-runs in Diamond and Dybvig (1983) model. The other source of the multiple solution is based on the interdependencies in the financial network. For instance, for any two banks \( i \) and \( j \), there might be two different \( v \) vectors satisfying equation (6) such that in one of these both \( i \) and \( j \) fail, and in the other both banks remain solvent.
The contagion algorithm below gives us the ultimate failures via the propagation of the bankruptcy costs in the network. The algorithm works as follows:

At step $t$ of the algorithm, let $\mathcal{N}_t$ be the set of failed banks. Initialize $\mathcal{N}_0 = \emptyset$. At step $t \geq 1$:

(i) Let $b_{t-1}$ be a vector with element $b_i = \beta_i$ if $i \in \mathcal{N}_{t-1}$ and $b_i = 0$ otherwise.

(ii) Let $\mathcal{N}_t$ be the set of all $k$ such that entry $k$ of the following vector is negative:

$$A[p - b_{t-1}] - l$$

(iii) Terminate if $\mathcal{N}_t = \mathcal{N}_{t-1}$. Otherwise return to step 1.

When the algorithm terminates at step $T$, the set $\mathcal{N}_T$ corresponds to the set of banks that fail in the best-case solution. The algorithm gives us the domino effects of failures in the network, where various banks default at each step that are triggered by the defaults in the previous steps. By using this algorithm one can find the set of failures for a given financial network $(C, F)$. This hierarchical default structure works for any given financial network.

Figure 1: Contagion in a financial network
Domino effects of failures are illustrated in Figure 1.\textsuperscript{18} A directed link from bank \(i\) to bank \(j\) implies that bank \(j\) is the creditor of bank \(i\), and there is a flow from bank \(i\) to bank \(j\) when the payments are realized. In the given illustration, bank 1 defaults after the shock hits, and the failure of bank 1 results in the failures of its creditor banks 2 and 4 in the second step. The failures of banks 1, 2, and 4 trigger the failure of bank 3 in the third step. Banks 5 and 6 remain solvent since these banks have no interbank claims in any of the failing banks.

**Definition 1.** Bank \(i\) is a distressed bank if \(i \in \mathbb{N}_T\), and it is a healthy bank if \(i \in N \setminus \mathbb{N}_T\).

Following the Definition 1, the red banks in Figure 1 are the distressed banks, and banks 5 and 6 are the healthy banks. I call the red banks the distressed banks rather than the failed banks, because the failure of these banks might be prevented by formation of rescue consortia, which is discussed in the next part. Definition 2 below follows from the contagion algorithm.

**Definition 2.** For any given financial network \((C, F)\), bank \(i\)’s distress rank is equal to \(dr_i = t\) if bank \(i\) defaults at step \(t\) of the contagion in the financial network \((C, F)\). \(dr_i = N\) for any bank \(i\) that does not default in \((C, F)\).

Figure 1 illustrates the distress ranks for each bank. The bank hit by the shock has distress rank equal to 1. Banks 2 and 4 that default in the second step of the contagion have distress rank equal to 2. Bank 3, which default at the third step of the algorithm, has distress rank equal to 3. Lastly, each healthy bank has distress rank equal to \(N\), which is equal to 6 in this illustration with 6 banks.\textsuperscript{19} In the next part, I introduce the bank rescues and discuss how the defaults that are illustrated here can be prevented.

\textsuperscript{18}The bank and the network characteristics are not specified in the illustration in Figure 1. A specific numerical example can always be written for this contagion case.

\textsuperscript{19}The distress rank for a healthy bank can be at most \(N\). In the extreme case, consider a network where only one bank defaults at each step of the algorithm and only one bank remains healthy at the end of the contagion. In that case, the distress rank will be \(N - 1\) for the bank that fails lastly and, thus, the distress rank will be \(N\) for the healthy bank. The distress rank is always less than or equal to \(N\) for a healthy bank in a given financial network, and that’s why I fix it to \(N\) for generality.
B. Rescues

First, I define the timing of the rescues in a given financial network. The shock hits a single bank’s proprietary asset and the rescue consortia are formed simultaneously, after the shock hits and before the payments are realized. A consortium might include two banks, referring to a rescue merger, or more than two banks, referring to a bail-in by multiple banks. \( \phi \subseteq N \) is the set of banks that are involved in any rescue consortium, and \( M = \{m_1, \ldots, m_n\} \) is the set of rescue consortia that are formed. In order to capture various rescue configurations, including the pairwise mergers and multi-bank bail-ins, the formation of rescue consortia is defined as a multi-bank merger environment, in which banks form merger(s) of two or more banks, where each bank is involved in at most one merger. A merger is defined in a standard way. The total assets and the total liabilities of the members of a given merger are summed up, and unchanged for the non-merged banks. Hence, the net worth of the banks involved in a merger is summed-up, and unchanged for the non-merged banks. A merger that is formed might be thought of as pooling the available capital for rescuing the distressed bank(s) in the given merger. For instance, the formation of the grand coalition represents a case such that all banks contribute to the rescue of the shocked bank at some degree, which is equivalent to recapitalizing the shocked bank via purchasing its shares.\(^{20}\)

On the other hand, rescue mergers do not necessarily include the bank hit by the shock. The rescue incentives and the formation of rescue consortia are discussed in detail in Section III. The initial results focus on the rescue capabilities in a given financial network. Formally, the mergers are defined as follows:

**Definition 3.** \((C, F)^M\) is the financial network after a set of banks \( \phi \subset N \) form the set of mergers \( M = \{m_1, \ldots, m_n\} \) in a given financial network \((C, F)\).

\(^{20}\)Pooling the assets and liabilities for each merger require a restructuring of the cross-holdings matrix. Lemma 2 in the Appendix shows that there exists a unique way of restructuring the cross-holdings which satisfies the properties of mergers in Definition 3 for every given proprietary asset return vector. The restructured cross-holdings are set according to the result in Lemma 2. Besides its uniqueness property, the restructuring rule given in Lemma 2 is a natural way of defining the restructured interbank obligations.
\((C, F)^M\) has the following properties:

i) \(N^M = (N \setminus \phi) \cup M\),

ii) \(p_j^M = p_j, l_j^M = l_j, \beta_j^M = \beta_j \forall j \in N \setminus \phi\),

iii) \(V_j^M = V_j, L_j^M = L_j \forall j \in N \setminus \phi\),

iv) \(p_{mk}^M = \sum_{k \in m_k} p_k, l_{mk}^M = \sum_{k \in m_k} l_k, \beta_{mk}^M = \sum_{k \in m_k} \beta_k \forall m_k \in M\),

v) \(V_{mk}^M = \sum_{k \in m_k} V_k, L_{mk}^M = \sum_{k \in m_k} L_k \forall m_k \in M\).

Next, I define the rescue merger. A merger \(m_k \in M\) is a \textit{rescue merger} if it prevents at least one additional failure compared to the case that \(m_k\) has not been formed, all else constant. Formally,

**Definition 4.** A merger \(m_k\) is a \textit{rescue merger} if \(\mathcal{N}_T^M \subset \mathcal{N}_T^{M \setminus m_k}\).

Definition 5 is on the resilience of a rescue merger.

**Definition 5.** The merger \(m_k\) in \((C, F)^M\) is at least as \textit{resilient} as the merger \(m_l\) in \((C', F')^{M'}\) if whenever \(m_k\) defaults in \((C, F)^M\), then \(m_l\) defaults in \((C', F')^{M'}\).

Proposition 1 is a result on the implications of the shock level and the bank characteristics in bank rescues.\(^{21}\)

**Proposition 1.** Consider a financial network \((C, F)\) in which a single bank, bank \(r\), is hit by the shock. Then,

\(^{21}\)The linearities in liabilities in the model might result in more than one initial failures following the shock that hits a single bank. Assumption 1 and 2 provided in the proof of Proposition 1 in the Appendix ensure that there is no such case, which makes the model equivalent to the standard models in that sense. Assumption 1 part \(i\) implies that the shock is high enough that the bank hit by the shock is always a distressed bank \((\overline{dr}_r = 1)\), and part \(ii\) implies that the only bank that would directly default after the shock is bank \(r\) \((\overline{dr}_i > 1\) holds for any bank \(i \neq r)\). Assumption 2 guarantees that if a merger including bank \(r\) is formed and prevents the failure of bank \(r\), then no other banks fail directly after the shock hits.
i) for any merger \( \{m_r : r \in m_r\} \), there exists a shock level \( s_{m_r}^* \in [0, p_r] \) such that the first failure is prevented by \( m_r \) and there exists no failure in \((C, F)^M\) iff \( s \leq s_{m_r}^* \). The threshold level \( s_{m_r}^* \) is non-increasing in the external liabilities \( (l_i) \) of each bank \( i \in m_r \).

ii) any merger \( m_k \in M \) becomes weakly more resilient by a weakly decrease in external liabilities \( (l) \) or bankruptcy costs \( (\beta) \).

Proposition 1 part i shows that there exists a threshold level of the shock that a given consortium including the bank hit by the shock can absorb. Moreover, the threshold level of the shock for such a consortium is non-increasing in the external liabilities of the banks involved in that consortium, since the shock absorption capacity of a given rescue consortium decreases as the external liabilities become higher. On the other hand, Proposition 1 part ii shows that any given consortium becomes less resilient when bankruptcy costs \( (\beta) \) or external liabilities \( (l) \) of the banks in the financial network are larger. The intuition behind this result is that the higher the external liabilities and the bankruptcy costs, all else constant, the higher the distress level in the financial system, which makes the rescue actions less effective. One can see that the level of the bankruptcy costs play no role in rescuing the shocked bank, but becomes crucial in preventing the contagious failures whenever the initial failure occurs.

C. The Network Characteristics and the Rescue Capabilities

In this part, I analyze the implications of the architecture of the financial network on the rescue capabilities. The diversification and the integration are the two measures of the network characteristics. The diversification level summarizes the information on how many counterparties each bank has, and the integration level summarizes the information on the level of the total interbank obligations of each bank in the network. Formally,

**Definition 6.** i) The financial network \((C', F)\) is more diversified than the financial network \((C, F)\) if and only if \( C'_{ji} \leq C_{ji} \) for all \((i, j)\) such that \( C_{ji} > 0 \), with strict inequality for some \((i, j)\), and \( C'_{ji} > C_{ji} = 0 \) for some \((i, j)\).
ii) The financial network \((C', F)\) is more integrated than the financial network \((C, F)\) if and only if \(\sum_{j:j \neq i} C'_{ji} \geq \sum_{j:j \neq i} C_{ji}\) for all \(i\), with strict inequality for some \(i\).

A financial network becomes more diversified when the number of creditors of each organization \(i\) weakly increases, where the interbank liabilities of bank \(i\) to its each original creditor weakly decrease. Thus, the diversification captures the spread of interbank contracts in a financial network. On the other hand, a financial network becomes more integrated if the ratio of the interbank liabilities to the total assets becomes higher for each bank. This refers to the higher ratio of interbank liabilities to total assets for each bank, which results in stronger interbank dependencies.

Next, I define a simplified environment for the network analysis. The banks in the financial network are identical in terms of asset sizes and the leverage ratios. Accordingly, I focus on \(d\)-regular networks in Sections II and III, and relax this assumption in Section IV, wherein I focus on the optimal interbank networks. Focusing on \(d\)-regular networks with homogeneity in bank size ensures that any inefficiency or the lack of rescue capabilities is due to the network structure rather than the heterogeneity in sizes or leverage ratios of the banks. The financial network is denoted by \((c, d, F)\) with the following properties:

\[
(c, d, F) := \begin{cases} 
|d_i^{in}| = |d_i^{out}| = d & \forall i \in N \\
C_{ii} = 0 & \forall i \in N \\
C_{ij} = \frac{c}{d} & \forall i, j : i \in d_j^{out} \\
C_{ij} = 0 & \forall i, j : i \notin d_j^{out} \\
\sum_{j \in N} C_{ji} = c < \frac{1}{2} & \forall i \in N \\
p_i = 1 & \forall i \in N \\
l_i = l \in (\frac{1}{2}, 1) & \forall i \in N
\end{cases}
\]

The ratio of the interbank liabilities to the total assets is equal to a constant
\[ c = \sum_{j \in N} C_{ji} < \frac{1}{2} \] for each bank \( i \in N \). Moreover, each bank have \( d \) number of creditors and borrowers. \( d^\text{out}_i \) is the set of creditors of bank \( i \), and \( d^\text{in}_i \) is the set of banks that borrow from bank \( i \). In addition, \( C_{ij} = \frac{c}{d} \) for all \((i, j)\) such that \( C_{ij} > 0 \), which capture the homogeneity in the weights of the links. In addition to this, I consider that the bankruptcy costs are such that \( p_i \) drops to zero if bank \( i \) defaults. Figure 2 below illustrates 2-regular networks. As shown in Figure 2, a \( d \)-regular network can be formed with different connectedness properties.

\[ \begin{align*}
\text{2-regular (disconnected)} & \quad \text{2-regular (connected)} \\
1 & \quad 1 \\
2 & \quad 6 \\
3 & \quad 2 \\
4 & \quad 5 \\
5 & \quad 3 \\
6 & \quad 4 \\
\end{align*} \]

**Figure 2:** Examples of \( d \)-regular networks: disconnected vs connected

Proposition 2 is on the existence of rescue mergers in financial networks.

**Definition 7.** A financial network is *contagious* if \( \{r\} \subsetneq \mathcal{N}_T \), where \( r \) is the bank hit by the shock and \( \mathcal{N}_T \) is the set of failures that would occur if no rescue consortia were formed. Any potential failure \( i \in \mathcal{N}_T \setminus r \) is called a contagious failure.

**Proposition 2.** Consider any \( d \)-regular network \((c, d, F)\) and a set of mergers \( M \) such that \( \overline{dr}_i = \overline{dr}_j \ \forall (i, j) \in m_k, m_k \in M \). Then, there exists no rescue merger \( m_k \in M \).
Proposition 2 highlights the fact that the existence of rescue consortia depends on the contagion structure and hence the architecture of the financial network. The result shows that a merger which only involves banks with same distress ranks, which is conditional on the location of the first failure, does not prevent any failure unless there exists any other merger which is formed by banks with different distress ranks. As an example, consider the complete network, in which each bank is a counterparty of all other banks in the network. Whenever the contagious failures occur in the complete network, all banks default simultaneously after the first failure. Proposition 2 implies that any consortium that does not involve the bank hit by the shock is not effective to prevent the contagious failures in the complete network, since all banks except the shocked bank have the same distress rank. Consequently, Proposition 1 and Proposition 2 together imply that for high levels of shock, whenever the rescue of the shocked bank becomes infeasible, an unstoppable contagion occurs in a contagious complete network. Proposition 3 below is the main result on the rescue capabilities in financial networks.

**Proposition 3.** Consider any contagious $d$-regular network $(c, d, F)$. Then,

i) for $N < N^*$ and $l > l^*$, there exist $\bar{c}$ and $\bar{d}$ such that if $c > \bar{c}$ or $d > \bar{d}$, then the contagious failures can only be prevented by rescuing the shocked bank,

ii) there exists no merger which can rescue the shocked bank if $s > N(1-l)$.

These together imply that the financial networks exhibit a “robust-yet-unpreventably fragile” nature.

The “robust-yet-fragile” nature of financial networks is a well-known result in the financial contagion literature, which implies that an increase in the diversification level increases the soundness of the system, yet many banks fail together if a problem occurs in the network. Following this insight, Proposition 3 shows that the financial networks exhibit a “robust-yet-unpreventably fragile” nature. The intuition behind this result is as follows. Contagious failures become imminent in a high diversified network if the integration level is sufficiently high. In that case, the high integration results in significant amount of losses for the rest of the system following the first failure. Accordingly, in
such networks, if the shock is sufficiently high, which makes the rescue of the
shocked bank infeasible, then the rescue of the rest of the system also becomes
infeasible due to the high losses following the first failure. For this reason,
the financial networks exhibit a “robust-yet-unpreventably fragile” nature. On
the other hand, part ii also implies that when the shock is large, the banks
can stop only the domino failures and maintain financial stability as long as
the integration level is sufficiently small. This result shows that under certain
conditions, a cascade of failures, if it is imminent, can be prevented via stop-
ning the contagious failures. Therefore, rescuing a bank that is hit by a large
shock is not necessary for maintaining the financial stability. Following the
main result on the rescue capabilities, a fundamental question arises: When
do the banks have incentives to rescue the distressed banks? In the next part,
I analyze the rescue incentives.

III. Rescue Incentives

In this part, I investigate the sources of inefficiencies in rescues. I introduce
a rescue consortia formation game to analyze the rescue incentives. Before
introducing the game, next, I define the social welfare.

**Definition 8.** The social welfare in \((c, d, F)^M\) is equal to
\[ W(c, d, F)^M = \left( \sum_{i \in N^M} p_i \right) - s \] if there exists no default in \((c, d, F)^M\), or equal to
\[ W(c, d, F)^M = \sum_{i \in N^M \setminus \{\mathbb{F}^M\}} p_i \] if there exist default(s) in \((c, d, F)^M\).

In the defined environment, maximizing the social welfare is equivalent to
minimizing the sum of the realized bankruptcy costs, which is equivalent to
minimizing the number of defaults, since the only losses in the network are the
shock, which is irreversible, and the potential bankruptcy costs. For any given
financial network \((c, d, F)\) and the shock \(s\) that hits a given single bank, a set
of mergers \(M\) is socially efficient if it maximizes the social welfare. Formally,

**Definition 9.** A set of mergers \(M\) is socially efficient if \(\nexists M' s.t. W(c, d, F)^M' > W(c, d, F)^M\).
Next, I define the rescue consortia formation game. The rescue consortia formation game is a simultaneous move game \( \Gamma = (N; (S_i)_{i \in N}; (f_i)_{i \in N}) \) consisting of set of banks \( N \), a strategy set \( S_i \) for each bank \( i \in N \), and a payoff function \( f_i : \prod_{i \in N} S_i \to \mathbb{R} \) for each bank \( i \in N \). The strategy set of bank \( i \) is \( S_i = \{ T \mid T \subset P(N) \} \) where \( P(N) \) is the set of subsets of \( N \). A particular strategy \( s_i \in S_i \) represents the set of mergers that bank \( i \) has willingness to be involved. Given the strategy profile \( s \), a merger \( m_k \) is formed if \( s_{jm_k} = 1 \) for all \( j \in m_k \) and there exists no other \( m_l \) and \( j \in m_k, m_l \) s.t. \( s_{im_l} = 1 \) for all \( i \in m_l \). If there exist ties, then the ties are broken in a way that the social welfare is maximized.

The solution concept is the coalition-proof Nash equilibrium. Let \( G \subset N \) be a coalition, and \( s_{N \setminus G}^* \in \prod_{i \in N \setminus G} S_i \). The game \( \Gamma = (N; (S_i)_{i \in N}; (f_i)_{i \in N}) \) induced on the players of \( G \) by the strategies \( s_{N \setminus G}^* \) has payoff functions \( f_i^* : \prod_{i \in N} S_i \to \mathbb{R} \) given by \( f_i^*(s_G) = f_i(s_G, s_{N \setminus G}^*) \) for all \( s_G \in \prod_{i \in G} S_i \), for each \( i \in G \). A strategy profile \( s^* \in \prod_{i \in N} S_i \) is self-enforcing if for all \( G \subset N \), it holds that \( s^*_G \) is a coalition-proof Nash equilibrium of the game \( \Gamma(s_{N \setminus G}^*) \). A strategy profile \( s^* \) is a coalition-proof Nash equilibrium of \( \Gamma \) if \( s^* \) is self enforcing and there is no other self enforcing strategy profile \( s \in S_N \) such that \( f_i(s) > f_i(s^*) \) for all \( i \in N \). In addition, the share parameters need to be defined for each potential merger \( m_k \subset P(N) \). For any given merger \( m_k \), \( \sum_{i \in m_k} \alpha_i^{m_k} = 1 \), where \( \alpha_i^{m_k} \) is the share of bank \( i \in m_k \). For any fixed sharing rule \( \alpha \), the payoffs to a player \( i \) from playing a strategy \( s_i \), given that other players choose \( s_{-i} \), is \( f_i(s_i, s_{-i}) = \alpha_i^{m_k} e_{m_k}^M \). I fix a sharing rule \( \alpha \) arbitrarily s.t. \( \sum_{i \in m_k} \alpha_i^{m_k} = 1 \) for any given \( m_k \subset P(N) \).

One source of inefficiencies in rescues is the disconnectedness in the network. Disconnectedness cause inefficiencies because the banks which are disconnected to the contagious region prefers not to contribute to the rescue of the banks in the contagious region since these banks have no potential losses from the failures. Proposition 4 and Corollary 1 show the inefficiencies due to the disconnectedness.

**Proposition 4.** Consider any contagious \( d \)-regular network \((c, d, F)\). Then,
there exists no coalition-proof Nash equilibrium in which a healthy bank is involved in a rescue merger.

Proposition 4 highlights the main insight into the rescue incentives. The banks which are least affected by the failure(s) have the least incentives to prevent the failure(s). The result shows that in a contagious $d$-regular network, healthy banks have no incentives for rescues. The result shows that in a contagious $d$-regular network, healthy banks have no incentives for rescues. This result holds under the assumption that the proprietary asset return of a failed bank drops to zero. For smaller bankruptcy costs, the main insight still holds: Bank rescues are costly, therefore a bank’s willingness to contribute to rescues increases as its potential losses rise. In this sense, the connectedness of the financial network gains importance, because the connectedness increases the extent of the contagious failures that incentivizes more banks to contribute in the rescue of the shocked bank. Corollary 1 highlights this fact and shows that if the shocked bank is located in a small-sized connected component, then, for sufficiently large shocks, the banks in the connected component are not able to prevent neither the first failure nor the contagious failures. In that case, the disconnectedness results in an unrescued local contagion.

**Corollary 1.** Consider a contagious $d$-regular network $(c,d,F)$ such that the shocked bank is in a distinct connected component of size $d + 1$, and the banks in $(c,d,F)$ have the capabilities to rescue the shocked bank or to stop the contagious failures (all failures but the first failure). Then, for $s > (d + 1)(1 - l)$, there exists no coalition-proof Nash equilibrium in which the shocked bank is rescued or the contagious failures (all failures but the first failure) are prevented. Consequently, an unprevented local contagion occurs in the connected component that the shocked bank is involved.

Figure 3 illustrates the result in Corollary 1, where the red banks represent the distressed banks in both configurations. In Figure 3, when the network is disconnected, banks 4, 5, and 6 have no incentives to prevent the failures in the connected component that the shocked bank is involved. This restricts the available capital for the rescue of the distressed banks. In addition, if the size
of the connected component including the bank that is hit by a large shock is sufficiently small, then a local contagion occurs. On the other hand, the available capital for rescues increase in the connected network, whereas the connectedness also brings a risk of system-wide cascade of failures whenever both the shock and the integration are sufficiently high. The ways to avoid such risks in connected networks are discussed in Section IV.

![Diagram showing contagion and rescue incentives in disconnected vs connected networks](image)

**Figure 3:** Contagion and rescue incentives in disconnected vs connected networks

In addition to the disconnectedness, another source of inefficiencies is the weak interbank dependencies in connected networks. The main insight here is that the higher the magnitude of the shock, the lower the incentives to rescue the shocked bank, since the leverage ratio of the shocked bank becomes too high after a large shock. This might still be the case even if there exists a potential cascade of failures that threatens all banks in the network. In this case, the banks use the alternative rescue method: rescuing the rest of the system instead of rescuing the shocked bank. Proposition 5 is the result on such selective rescue actions. Before Proposition 5, I provide a simplifying assumption for the rescue consortia formation game.
In the defined environment with externalities and complementarities in rescue actions, the Nash equilibrium of this game might not be unique; hence there might exist multiple coalition-proof Nash equilibria as well, depending on the sharing rule. Moreover, the coalition-proof Nash equilibria might not even exist, which again depends on the sharing rule. However, in a contagious connected $d$-regular network, we expect to see formation of some rescue consortia as long as it is feasible, because otherwise all banks would default. One way to incorporate the arbitrary sharing rules into the game is limiting the set of strategies of the banks. I restrict the set of strategies as follows. The strategy set of bank $i$ is $S_i = \{ T \mid T \subset \{ i, (N \setminus r), N \} \}$.

The sharing rule under the restricted set of strategies determines the contribution of each bank in a given rescue consortium. In the remaining part of the paper, I use the restricted set of strategies and keep using an arbitrarily selected sharing rule. First, this restriction on the set of strategies enables me to omit the inefficiencies caused by the sharing rule itself. Second, this restriction allows me to focus on the selective rescue decisions of the banks, independently from the complications due to the potential multiplicity of equilibria caused by the multiplicity of the set of rescue mergers. Therefore, allowing arbitrary sharing rules under the restrictions on the set of strategies might be thought of as banks’ collective decisions on purchasing a certain amount of the shares of the shocked bank or its creditors under risk, where each bank’s contribution is determined by the given sharing rule. Proposition 5 shows the banks’ selective rescue decisions depending on the level of the shock.

**Proposition 5.** Consider a contagious, connected $d$-regular network $(c, d, F)$ such that the banks in $(c, d, F)$ have the capabilities to rescue the shocked bank or to stop the contagious failures (all failures but the first failure), call it $\psi_d$. Fix any sharing rule $\alpha$ such that the shareholders of the shocked bank receives nothing whenever the shocked bank is rescued ($\alpha_{r \{ N \}} = \alpha_{r \{(N \setminus r), r \}} = 0$). Then, there exists an $\bar{s}$ such that:

i) for $s \leq \bar{s}$, there exists a coalition-proof Nash equilibrium in which the shocked bank is rescued,

ii) for $s > \bar{s}$, there exists a coalition-proof Nash equilibrium in which the

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Proposition 5 explains the inefficiencies that arise under a high shock regime.\footnote{For simplicity, I fixed the sharing rule such that $\alpha^{(N)}_r = 0$. The result holds for small enough $\alpha^{(N)}_r$.} The result also implies that the inefficiencies can potentially be avoided by increasing the level of the integration. The main insight here is that if the integration is sufficiently high, then the remaining banks find it more profitable to directly rescue the shocked bank instead of rescuing only the rest of the system. The reason is that the higher the level of the integration, the higher the losses due to the failure of the shocked bank, which makes the failure of the shocked bank too costly for the system. As a result, the inefficiencies in the rescue of the shocked bank might be eliminated by increasing the level of the potential losses, so the integration, up to a threshold level, which is one of the main insights into the optimal networks analysis.

IV. Optimal Interbank Networks

In this part, I provide results on the network characteristics which would potentially overcome the inefficiencies in rescues. In this part, first, I redefine the shock environment. Different from the previous parts, there exists a random shock $s$, which hits a bank uniformly at random and can be either small or large with the following probabilities:

\[
s := \begin{cases} \text{s}_S = N(1 - l) - \epsilon < 1 & \text{with probability q} \\ 1 > s_L > N(1 - l) & \text{with probability } 1 - q \end{cases}
\]

The small shock $s_S$ ensures that only the grand coalition can prevent the first failure. The reason for assigning the worst-case level for the small shock is that in some cases even if the first failure is preventable, the shock might be too high that the grand coalition might be required to stop the first failure. On the other hand, the large shock $s_L$ ensures that there exist no consortium which can prevent the initial failure. Both conditions require that there...
exists sufficiently small number of banks that have sufficiently large external liabilities. In addition to the standard assumption that $\sum_{j \in N} C_{ji} < \frac{1}{2}$ for all $i$, I consider that $N(1 - l) < \frac{1}{2}$ holds.

Next, I define a class $\Omega$ of interbank networks. A financial network $(C, F) \in \Omega$ is formed by the set of banks $N = \{1, ..., n\}$ and has the following features:

$$\psi(C, F) \in \Omega := \left\{ \begin{array}{ll}
C_{ii} = 0 & \forall i \in N \\
C_{ij} \geq 0 & \forall i, j \in N \\
\sum_{j \in N} C_{ji} < \frac{1}{2} & \forall i \in N \\
p_i = 1 & \forall i \in N \\
l_i = l \in \left( \frac{1}{2}, 1 \right) & \forall i \in N \\
N(1 - l) < \frac{1}{2}
\end{array} \right. $$

In summary, any $\psi(C, F), \psi'(C', F) \in \Omega$ have the same bank characteristics and can differ only in the network characteristics. The homogeneity in proprietary asset size and the external liability size is an assumption for tractability, and the main insight holds also for heterogeneous size of banks. In addition to this, $p_i$ drops to zero if bank $i$ defaults for all $i \in \psi(C, F)$, for any given $\psi(C, F) \in \Omega$. Similar to the previous part, I restrict the set of strategies such that $S_i = \{T \mid T \subset \{i, (N \setminus r), N\}\}$. The sharing rule is determined after the shock hits a bank uniformly at random and before the formation of rescue consortium. I fix a sharing rule $\alpha$ arbitrarily, which satisfies $\sum_{i \in m_k} \alpha_{i}^{mk} = 1$ for all $m_k \subset \{i, (N \setminus r), N\}$ and $\alpha_r^{\{N\}} = 0$ for bank $r$ that is hit by the shock.

Regarding the linkages, the class $\Omega$ is a large class of networks. The diversification and the integration levels might differ among banks in a given $\psi(C, F) \in \Omega$, and hence the network characteristics might be different for any given $\psi(C, F), \psi'(C', F) \in \Omega$. In summary, there exists no restriction on the architecture of the network in the defined class $\Omega$ of financial networks. The only restriction is the standard assumption that $\sum_{j \in N} C_{ji} < \frac{1}{2}$ for all $i$, for any given $\psi(C, F) \in \Omega$. Given these specifications, the constrained optimization problem is as follows:

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\[
\max_{\psi(C,F)} W(\psi(C,F)^{M^*_\psi(C,F)}) \text{ subject to } \psi(C,F) \in \Omega
\]

where \(M^*_\psi(C,F)\) is the set of mergers in the best-case CPNE in \(\psi(C,F)\) in which as few banks as possible fail. Next, I formally define the optimality condition.

**Definition 10.** An interbank network \(\psi^*(C,F) \in \Omega\) is optimal if

\[
W(\psi^*(C,F)^{M^*_\psi^*(C,F)}) \geq W(\psi(C,F)^{M^*_\psi(C,F)})
\]

for every interbank network \(\psi(C,F) \in \Omega\).

Following the Definition 10, if there exists any \(\psi(C,F)\) with the following properties, then \(\psi(C,F)\) is an optimal interbank network:

1. whenever the rescue of the shocked bank is feasible, which is the case when \(s = s_S\), then the shocked bank is rescued in \(\psi(C,F)\).
2. whenever the rescue of the shocked bank is infeasible, which is the case when \(s = s_L\), then either there exists no contagious failures in \(\psi(C,F)\), or the contagious failures, if exist, are always stopped in \(\psi(C,F)\).

The reason why such an interbank network is optimal is that it eliminates the bankruptcy costs as much as possible. Therefore, in such a network, the self-correcting rescue mechanism works in the most effective way and thus the government’s role in rescues is minimized. Before I provide the result on the optimal networks, I define a class of \(d\)-regular networks: \(d\)-ring lattices.

**Definition 11.** A \(d\)-ring lattice \((c, d^R, F)\) has the following properties:

\[
(c, d^R, F) := \begin{cases}
C_{ii} = 0 & \forall i \in N \\
C_{ij} = \frac{c}{d} & \forall \{i, j : i - d \leq j \leq i - 1\} \\
C_{ij} = 0 & \text{otherwise}
\end{cases}
\]

Figure 4 is an illustration of a sequence of \(d\)-ring lattice, in which each bank is the creditor of the closest \(d\) number of preceding banks and the borrower of the closest \(d\) number of following banks for any given \(d\), where \(d = 1\) is the
ring network, and $d = N - 1$ is the complete network in which every pair of distinct nodes is connected by a pair of links. Proposition 6 provides a class of optimal networks.

**Proposition 6.** There exist $c, \overline{c}$ and $\overline{d}$ such that any $d$-ring lattice $(c, d^R, F) \in \Omega$ which satisfies:

i) $1 - A_r = (N - 1)(1 - l)$  

ii) $c \leq c \leq \overline{c}$ (integration is intermediate)  

iii) $d \leq \overline{d} < \frac{(1-s)^2}{1-l}$ (diversification is low)  

is an optimal network. There exists an optimal $d$-ring lattice $(c, d^R, F) \in \Omega$ which satisfies the properties above.

The class of optimal networks given in Proposition 6 has the following features:

1. The financial network is evenly connected with no clustering/disconnectedness.

2. The diversification in the evenly connected network is low enough that even for low integration levels, a risk of a system-wide cascade of failures emerges, which maximize the number of banks that have willingness to contribute in the rescue of the shocked bank.

3. The integration level is sufficiently low that the banks can stop the contagious failures whenever the shocked bank is non-rescuable for large shocks ($s = s_L$).

4. The integration level is sufficiently high that the banks directly rescue the shocked bank whenever it is rescuable, which is the case if $s = s_S$. 


The main insight into this result is as follows. Designing a financial network in which the potential domino effects pose bankruptcy risks for all banks, maximize the available contributions in the rescue of the shocked bank. This is satisfied if the financial network is evenly connected, which eliminates any clustering or disconnectedness. The required level of the integration to create such a systemic risk increases by an increase in the diversification level. Besides, the previous results show that the financial networks exhibit “robust-yet-unpreventably fragile” nature. This implies that in dense networks, creating such a systemic risk makes the financial system unpreventably-fragile. On the
other hand, the required level of the integration that poses such a systemic risk in a sufficiently low diversified and evenly connected network is small enough that the financial system is not unpreventably-fragile. Hence, in low diversified and evenly connected networks, having sufficiently high integration pose i) a system-wide contagion risk that maximizes banks’ contributions in rescue, ii) high amounts of potential losses to the counterparties, which ensure that banks directly rescue the shocked bank instead of stopping only the domino failures. Moreover, having sufficiently low integration ensures that the potential losses of the counterparties due to the failure of the shocked bank are small enough that the remaining banks can always prevent the contagious failures whenever the shock is large and rescuing the shock bank is infeasible. As a result, low diversification and intermediate integration per bank accompanied by a well-designed architecture of the network with no clustering becomes the preferred network structure.

Proposition 7. Any non-contagious d-ring lattice \((c, d^R, F) \in \Omega\) is a non-optimal network.

Proposition 7 shows the inefficiencies that arise in the rescue of the shocked bank in non-contagious networks. The main insight here is that in non-contagious networks, for large enough shocks, banks prefer not to rescue the shocked bank since the rescue becomes relatively costly. Next, I discuss the government intervention in bank rescues.

V. The Government Assistance in the Bank Rescues

First, I revisit the rescue consortia formation game. At the beginning of the game, the government announces the amount of its assistance for each possible merger, again under the restricted strategies. Then, the payoffs in the revised game is \(f_i(s_i, s_{-i}) = \alpha_i m^k_i(c^M_{mk} + t^g_{mk})\). At the end of the game, if there exists a merger such that \(t^g_{mk} > 0\) and the banks have agreed on forming that merger, then the merger is formed with the promised government transfer. The social welfare with the government intervention is redefined as follows:
Definition 12. The social welfare in \((c, d, F)^M\) with the government assistance is equal to \(W(c, d, F)^M = \left( \sum_{i \in N^M} p_i \right) - s - \sum_{m_k \in N^M} t_{m_k}\) if there exists no default in \((c, d, F)^M\), or equal to \(W(c, d, F)^M = \sum_{i \in N^M \setminus \{m\}} p_i - \sum_{m_k \in N^M} t_{m_k}\) if there exist default(s) in \((c, d, F)^M\).

Under the restricted set of strategies, \(m_k\) is a singleton and either the grand coalition or the coalition of all banks but the shocked bank. Proposition 8 shows the selective rescue decision of the government. When the shock is sufficiently large, the government prefers to assist in the rescue of the rest of the system instead of the rescue of the shocked bank. This result together with the previous results offers explanations for the rescues during the 2007-2008 financial crisis, which excluded the banks hit by large shocks and saved the rest of the system via the formation of government-assisted rescue mergers.

Proposition 8. Consider a connected, contagious \(d\)-regular network \((c, d, F)\) such that rescuing the shocked bank or preventing the contagious failures (all failures but the first failure) are infeasible without the government-assistance. Then, there exists an \(s^*\) such that

i) for \(s \leq s^*\), the shocked bank is rescued by a government-assisted rescue consortium,

ii) for \(s > s^*\), the government does not assist in the rescue of the shocked bank, yet the contagious failures (all failures but the first failure) are prevented by a government-assisted rescue consortium,

where \(s^*\) is increasing in the integration level \(c\), which implies that the government is more likely to assist in the rescue of the shocked bank as the ratio of the “interbank liabilities of the shocked bank to the shock level” increases.

On the other hand, in an optimal network, one can see that there is no need for the government assistance when the shock is small, and any optimal network minimizes the bail-out costs to the government since the contributions in the rescue of the shocked bank is maximized in an optimal network.
Then, the question is that when does the government assists in the rescue of the shocked bank? The government assists when it is welfare enhancing. Proposition 9 shows that in an optimal network, the government assists in the rescue of the shocked bank if the shock is small enough. For Proposition 9, I consider the environment that is defined in Section IV, wherein I analyze the optimal networks.

**Proposition 9.** Consider an optimal network \((c, d^R, F) \in \Omega\). Then,

i) for \(s = s_S\), the shocked bank is rescued by a non-assisted rescue consortium.

ii) \(s = s_L \leq \frac{1 + N(1 - l)}{2}\), the shocked bank is rescued by a government-assisted rescue consortium.

iii) for \(s = s_L > \frac{1 + N(1 - l)}{2}\), the government does not assist in the rescue of the shocked bank, yet the contagious failures (all failures but the first failure) are prevented by a non-assisted rescue consortium.

This last result shows that the government’s rescue decisions are based on the “ratio of the shock to the potential losses” due to the failure of the shocked bank. As this ratio increases, the rescue of the shocked bank becomes too costly, and the government decides not to intervene in the rescue of the shocked bank.

**VI. Conclusion**

There is a historical debate over the bank rescues and financial stability. Lastly, the 2007-2008 global financial crisis has shown that the early diagnosis and treatment of a financial crisis play a key role in maintaining stability of an interconnected financial system. On the other hand, the costly bail-outs during the 2007-2008 financial crisis brought the private-sector solutions into the forefront of the discussions. In this paper, I develop a framework to analyze the bank rescues. The rescue analysis is incorporated into a financial network model, which provides new insights into the bank rescues. A costly bankruptcy
causes defaults on interbank debt and increases the systemic risk, which creates incentives for rescues. However, rescues are costly for the rescuer banks, and there exist complementarities in rescue actions. The complementarities in actions together with the negative externalities explain the public goods game nature of bank rescues.\footnote{For surveys on games on networks, see Jackson (2008) and Jackson and Zenou (2015).} In this sense, the architecture of the interbank network determines the extent of not only the potential bank failures but also the bank rescue decisions.

In this paper, I show that the rescue mechanism without government intervention works most effectively in an evenly connected network (with no clustering) in which each bank borrows/lends at intermediate levels from/to a small number of banks. These specifications can be explained briefly as follows. In such networks, the potential total losses in the financial system due to an initial failure are high enough that the other banks are maximally incentivized for the rescue of an initially distressed bank. On the other hand, the potential losses of the immediate counterparties of an initially distressed bank are low enough that whenever the rescue of the initially distressed bank becomes infeasible (e.g. high shock regime), then the banks can always stop the potential domino failures and maintain financial stability.

From another perspective, optimal interbank networks can be thought of as an environment in which the self-correcting mechanism of the financial network works most effectively. The policy implications of this paper lie in the interplay between the banks’ and the government’s rescue decisions. Accordingly, the results provide policy recommendations on the bank level limits of the interbank exposures as well as the system-wide design of the interbank network. The results imply that the effectiveness of the private sector solutions, which minimizes the government’s role in bank rescues during distressed times requires a system-wide design of the interbank linkages in the pre-crisis period. In this regard, the next step of research is investigating the formation of the interbank networks, in which banks consider the potential bank rescue actions in distressed times.

Finally, the results provide a rationale for why some banks (e.g. Lehman
Brothers, Washington Mutual) were excluded from the government-supported rescues in 2007-2008. I show that the rescues, which are designed to maintain financial stability, might exclude a bank hit by a large shock, since the rescue of such a bank becomes relatively costly if the ratio of the shock to the potential losses is too high. In that case, stopping the potential domino failures becomes the preferred method. Therefore, the results provide a rationale for the selective rescue decisions in 2007-2008.

References


Appendix

Lemma 1 below is useful for the contagion analysis.

**Lemma 1.** i) $A_{ij} = \sum_{k \in N} A_{ik} C_{kj}$ holds for all $i \neq j$.

ii) $A_{ii} = \bar{C}_{ii} + \sum_{k \in N} A_{ik} C_{ki}$ holds for all $i$.  

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PROOF OF LEMMA 1:

First, I show that $A = AC + \hat{C}$ holds for any given $C$ matrix such that $\hat{C}$ is a diagonal matrix with entries $\hat{C}_{ii} = 1 - \sum_{j \in N} C_{ji} > 0$ for all $i$ and $\hat{C}_{ij} = 0$ for all $i \neq j$, and $A = \hat{C}(I - C)^{-1}$.

Suppose that $A = AC + \hat{C}$ holds. Then,

$A = AC + \hat{C}$ implies

$\hat{C}(I - C)^{-1} = \hat{C}(I - C)^{-1}C + \hat{C}$, which can be rewritten as:

$\hat{C}(I - C)^{-1} = \hat{C}[(I - C)^{-1}C + I]$.

Since $\hat{C}$ is a diagonal matrix,

$(I - C)^{-1} = [(I - C)^{-1}C + I]$ holds, which can be rewritten as:

$(I - C)^{-1}(I - C) = I$, and hence $I = I$ holds, which completes the proof.

$A = AC + \hat{C}$ implies $A_{ii} = \hat{C}_{ii} + \sum_{k \in N} A_{ik}C_{ki}$ holds for all $i$, and $A_{ij} = \sum_{k \in N} A_{ik}C_{kj}$ holds for all $i \neq j$.

end of proof.

Lemma 2. Consider a financial network $(C, F)$ in which a set of banks $\phi \subset N$ form the set of mergers $M = \{m_1, ..., m_n\}$, which satisfies the properties below:

$N^M = (N \setminus \phi) \cup M$

$p_j^M = p_j \quad l_j^M = l_j \quad \beta_j^M = \beta_j \quad \forall j \in N \setminus \phi$

$p_{m_k}^M = \sum_{k \in m_k} p_k \quad l_{m_k}^M = \sum_{k \in m_k} l_k \quad \beta_{m_k}^M = \sum_{k \in m_k} \beta_k \quad \forall m_k \in M$

Then, for any given set $M$, there exist unique structures for $C^M$ and $A^M$ that satisfy

$V_j^M = V_j \quad L_j^M = L_j \quad \forall j \in N \setminus \phi$

$V_{m_k}^M = \sum_{k \in m_k} V_k \quad L_{m_k}^M = \sum_{k \in m_k} L_k \quad \forall m_k \in M$

for every asset return vector $p \in \mathbb{R}_+^N$. The unique structures have the following properties:

$C_{ij}^M = C_{ij} \quad \forall i, j \in N \setminus \phi$

$C_{m_k}^M = \left(\frac{\sum_{k \in m_k} \sum_{l \in m_l} (C_{lk}V_l)}{\sum_{l \in m_l} V_l}\right) \quad \forall m_k \in M$

$C_{m_k m_l}^M = \left(\frac{\sum_{l \in m_l} (C_{lk}V_l)}{\sum_{l \in m_l} V_l}\right) \quad \forall m_k, m_l \in M$

$C_{m_k}^M = \left(\frac{\sum_{k \in m_k} C_{kj}}{\sum_{k \in m_k} V_k}\right) \quad \forall j \in N \setminus \phi$

$C_{jm_k}^M = \left(\frac{\sum_{k \in m_k} (C_{jk}V_k)}{\sum_{k \in m_k} V_k}\right) \quad \forall j \in N \setminus \phi$
the new cross-holdings
this, I rewrite the equations after the merger by using the asset returns and
are satisfied after the merger:

\[ k \]  

\[ \text{The restructuring of the cross-holdings can be summarized as follows:} \]

i) the cross-holdings among the non-merged banks remain same,

ii) the cross-holdings that a given merger holds in a given non-merged bank

is equal to the sum of the cross-holdings that each bank in that merger holds

in that non-merged bank,

iii) the cross-holdings of a given non-merged bank in a merger is equal to

the weighted sum of its cross-holdings in each bank in the merger,

iv) the cross-holdings of a given merger in another given merger is a

weighted sum of the cross-holdings of each bank in the banks in the other

merger.

**Proof of Lemma 2:**

The equations below are satisfied before the mergers:

\[ V_j = (\sum_{k \in \phi} C_{jk} V_k) + (\sum_{i \in N \setminus \phi} C_{ji} V_i) + p_j \]

\[ \sum_{k \in m_k} V_k = (\sum_{k \in m_j \in N \setminus \phi} C_{kj} V_j) + \sum_{k \in m_k} p_k \]

Call the cross-holdings matrix given in Lemma 2 \( C^* \). The equations below

are satisfied after the merger:

\[ V_j^M = (\sum_{i \in N \setminus \phi} C_{ji}^* V_i^M) + \sum_{m_k \in M} (C_{jm_k}^* V_{m_k}^M) + p_j^M \forall j \in N \setminus \phi \]

\[ V_{m_k}^M = (\sum_{j \in N \setminus \phi} C_{mkj}^* V_j^M) + C_{mk m_k}^* V_{m_k}^M + (\sum_{m_l \neq m_k} C_{mk m_l}^* V_{m_l}^M) + p_{m_k}^M \forall m_k \in M \]

I claim that \( C^* \) satisfies the given properties in Lemma 2. In order to show

this, I rewrite the equations after the merger by using the asset returns and

the new cross-holdings \( C^* \) after the merger, which are given in Definition 3:

\[ V_j^M = (\sum_{i \in N \setminus \phi} C_{ji}^* V_i^M) + \sum_{m_k \in M} \left( \frac{\sum_{k \in m_k} (C_{jk} V_k)}{\sum_{k \in m_k} V_k} V_{m_k}^M \right) + p_j \forall j \in N \setminus \phi \]

\[ V_{m_k}^M = (\sum_{j \in N \setminus \phi} C_{kj}^* V_j^M) + \left( \sum_{k \in m_k} \frac{\sum_{l \in m_k} (C_{lk} V_l)}{\sum_{k \in m_k} V_k} V_{m_k}^M \right) V_{m_k}^M + \sum_{m_l \neq m_k} \left( \frac{\sum_{l \in m_l} C_{lk} V_l}{\sum_{l \in m_l} V_l} \right) V_{m_l}^M + \sum_{k \in m_k} p_k \]
We know that the financial system before the merger has a unique solution. We also know that the new system has a unique solution. Then, if we plug $V_j = V^M_j \forall j \in N \setminus \phi$ and $V^{M}_{m_k} = \sum_{k \in m_k} V_k \forall m_k \in M$ into the equations for $V^M_j$ and $V^M_{m_k}$, we get:

\[ V_j = \left( \sum_{i \in N \setminus \phi} C_{ji} v_i \right) + \sum_{m_k \in M} \left( \sum_{k \in m_k} C_{jk} v_k \right) + p_j = \left( \sum_{k \in \phi} C_{jk} v_k \right) + \left( \sum_{i \in N \setminus \phi} C_{ji} v_i \right) + p_j \]

and

\[ \sum_{k \in m_k} V_k = \left( \sum_{k \in m_k, j \in N} C_{kj} V_j \right) + \sum_{k \in m_k} p_k \]

These are the equations that we had before the mergers. Thus, from the uniqueness property, $V_j = V^M_j \forall j \in N \setminus \phi$ and $V^{M}_{m_k} = \sum_{k \in m_k} V_k \forall m_k \in M$ is also the unique solution for the system after the merger. So, $C^*$ satisfies the properties given in Lemma 2.

Next, I show that $A^*$ also satisfies the properties given in Lemma 2. The given properties in Definition 1 implies that $e_j = e^M_j \forall j \in N \setminus \phi$ and $e^M_m = \sum_{k \in m_k} v_j \forall m_k \in M$ holds for the shareholders’ equity. Equation (6) implies $v = Ap$ holds before the shock, and hence $e = v - l = Ap - l$ holds before the shock. Since $l$ is given exogenously, $v = v^M$ must hold, and hence $A^M p^M = Ap$ must hold.

For any given $(C, F)$, $A$ is the unique Leontief inverse matrix derived from the given $C$ matrix. By Definition 1, for a given set of mergers, one can find $p^M$ and $v^M$. Next, I show that $v^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$, which means that $A^*$ satisfies the given properties in Lemma 2.

\[ v^M_j = \sum_{m_k \in M} (A^*_{j m_k} p^M_{m_k}) + \sum_{i \in N \setminus \phi} A^*_{ji} p^M_i = \sum_{m_k \in M} \left( \frac{\sum_{k \in m_k} (A_{jk} p_k)}{\sum_{k \in m_k} p_k} \right) \sum_{k \in m_k} p_k + \sum_{i \in N \setminus \phi} (A^*_{ji} p_i) = \sum_{i \in N} A_{ji} p_i = v^M_j \]

holds for every $p \in \mathbb{R}^N_+$. Similarly,

\[ v^M_{m_k} = A^*_{m_k m_k} p^M_{m_k} + \sum_{m_l \neq m_k} (A^*_{m_k m_l} p^M_{m_l}) + \sum_{j \in N \setminus \phi} A^*_{m_k j} p^M_j = \left( \frac{\sum_{k \in m_k} \sum_{l \in m_l} (A_{kl} p_l)}{\sum_{k \in m_k} p_k} \right) \sum_{k \in m_k} p_k + \sum_{m_l \neq m_k} \left( \frac{\sum_{l \in m_l} \sum_{k \in m_k} (A_{kl} p_l)}{\sum_{l \in m_l} p_l} \right) \sum_{l \in m_l} p_l + \sum_{j \in N \setminus \phi} \sum_{k \in m_k} A_{kj} p_k = \sum_{k \in m_k, j \in N} A_{kj} p_j = \sum_{k \in m_k} v^M_k \]

holds for every $p \in \mathbb{R}^N_+$. Thus, $v^M = A^* p^M$ holds for every $p \in \mathbb{R}^N_+$. This completes the first part of the proof that $C^M$ and $A^M$ satisfy the given
properties of mergers for every \( p \in \mathbb{R}^N_+ \).

Next, I show that there exist unique structures for \( C^M \) and \( A^M \) satisfying the desired properties for every asset return vector \( p \in \mathbb{R}^N_+ \).

First, I show that \( A^* \) is the unique structure satisfying the properties in Lemma 2 for every \( p \in \mathbb{R}^N_+ \).

I already showed that \( v^M = A^*p^M \) holds for every \( p \in \mathbb{R}^N_+ \) for any given \( M \). Fix any given \( M \). Suppose that there exist another matrix \( A^{**} \) satisfying the desired properties on total assets and liabilities for every \( p \in \mathbb{R}^N_+ \) for \( M \). Then, \( v^M = A^{**}p^M \) holds for every \( p \in \mathbb{R}^N_+ \). Then, \( A^{**}\epsilon_j = A^*\epsilon_j \forall \epsilon_j \) holds where \( \epsilon_j \) is the vector such that \( j^{th} \) element of \( \epsilon_j \) is equal to 1 and all other elements of \( \epsilon_j \) are equal to \( \epsilon \).

\[ A^{**}\epsilon_j = A^*\epsilon_j \forall \epsilon_j \] implies \( A^{**}_{kj} = A^*_{kj} \forall k, j \) as \( \epsilon \to 0 \). Then, this implies \( A^* = A^{**} \). Thus, \( A^* \) is unique.

Next, I show the uniqueness of \( C^* \). I already showed that \( V^M = (I - C^*)^{-1}p^M \) holds for every \( p \in \mathbb{R}^N_+ \) for any given \( M \). Suppose that there is another \( C^{**} \) satisfying \( V^M = (I - C^{**})^{-1}p^M \) for every \( p \in \mathbb{R}^N_+ \) for \( M \). Then, similarly, \( (I - C^*)^{-1} = (I - C^{**})^{-1} \) must hold.

\( (I - C^*)^{-1} = (I - C^{**})^{-1} \implies ((I - C^*)^{-1})^{-1} = ((I - C^{**})^{-1})^{-1} \implies I - C^* = I - C^{**} \) holds from the uniqueness of the inverse matrix, which also implies \( C^* = C^{**} \). Thus, \( C^* \) is unique.

Lastly, \( A^* \) and \( C^* \) being the unique structures satisfying the desired properties for every \( p \in \mathbb{R}^N_+ \) for any given \( M \) implies that \( A^* = \hat{C}^*(I - C^*)^{-1} \), where \( \hat{C}^* \) is a diagonal matrix derived from \( C^* \).

Thus, \( A^M \) and \( C^M \) are the unique structures satisfying the desired properties for every asset return vector for any given set of mergers, which completes the proof.

**Proof of Proposition 1:**

**Assumption 1.**

\[ i) \frac{v_i - l_i}{A_{rr}} < s, \]
\[ ii) \frac{v_i - l_i}{A_{ir}} \geq s \forall i \neq r. \]
Assumption 2. \( i \) \( \frac{v_i - l_i}{\sum_{k \in I_r} (A_{ik}p_k)} \geq s \) for \( \{ m_r \mid r \in m_r \} \) and \( \forall i \notin \phi \).

\( ii \) \( \frac{\sum_{i \in m} (v_i - l_i)}{\sum_{k \in I_r} (\sum_{l \in m} (A_{ik}p_k))} \geq s \) for \( \{ m_r \mid r \in m_r \} \) and \( \forall m_l \neq m_r \).

By Assumption 1, \( e_i = v_i - l_i - sA_{ir} \geq 0 \) \( \forall i \neq r \). Consider any given set of mergers \( M \) in which there exists a merger \( m_r \) which involves bank \( r \) and prevents the failure of bank \( r \). Call the set of banks in \( m_r \) set \( R \). Then, Assumption 2 guarantees that any merger formed among the banks in the set of banks \( N \setminus R \) or any singleton in \( N \setminus R \) remains solvent. Thus, the contagion algorithm implies that there exists no other failure if there exists a merger which prevent the default of the bank hit by the shock. Conditional on this fact, merger \( m_r \) does not default iff \( e^M_{m_r} = \sum_{k \in I_r} (v_k - l_k) - sA_{m_r m_r} \geq 0 \). Thus, \( m_r \) does not default iff \( s \leq s^*_{m_r} = \frac{\sum_{k \in I_r} (v_k - l_k)}{\sum_{k \in I_r} (\sum_{l \in m} (A_{ik}p_k))} \).

For any given network \( (C, F) \), \( v = Ap \), where \( A = \hat{C}(I - C)^{-1} \). For any given \( C \) and \( p \), fix each \( v_i = \bar{v}_i \), and for any given \( \{ m_r \mid r \in m_r \} \) fix \( A_{m_r m_r} = \bar{A}_{m_r m_r} \), and weakly increase external liabilities \( l_i \) for all \( i \in m_r \), and call it \( l'_i \).

Then, the threshold level of shock becomes \( s^*_{m_r} = \frac{\sum_{k \in I_r} (v_k - l'_k)}{A_{m_r m_r}} \leq \frac{\sum_{k \in I_r} (v_k - l_k)}{A_{m_r m_r}} = s^*_{m_r} \).

Hence, the threshold level \( s^*_{m_r} \) is non-increasing in the external liabilities of each bank \( i \in m_r \).

**Proof of Part \( ii \)**

For any given \( (C, F) \) of set \( N \) of banks, index banks and form a given set of mergers \( M \) and reach \( (C, F)^M \). Next, consider another network \( (C', F') \) of set \( N \) of banks such that \( C = C' \), \( p = p' \), \( l_i \geq l'_i \) and \( \beta_i \geq \beta'_i \). For the given \((C', F')\), by using the same index for banks, form the same set of mergers \( M \) and reach \((C', F')^M\). The dependency matrices are equal in both networks, \( A = A' \), which implies that \( v_i (C, F) = v_i (C', F') \).

Fix \( v_i (C, F) = v_i (C', F') = \bar{v}_i \), and \( A_{ij} = A'_{ij} = \bar{A}_{ij} \) in both networks.

Proposition 1 part \( i \) implies that if there exists a merger which involves
bank \( r \), then \( m_r \) is weakly more resilient in \((C', F')^M\) than in \((C, F)^M\). By Assumption 2, if \( m_r \) is solvent in both networks, then there exists no failure in both networks. If \( m_r \) defaults in \((C, F)^M\) but remains solvent in \((C', F')^M\), then for any \( i \neq m_r \in (C, F)^M \), \( e_i(C, F)^M = \bar{v}_i - l_i - (\beta'_{m_r} + s)\bar{A}_{im_r} \), and it is strictly lower than \( e'_i(C', F')^M = \bar{v}_i - l'_i - (\beta'_{m_r} + s)\bar{A}_{im_r} \). Thus, in this case any merger in \((C', F')^M\) is strictly more resilient than in \((C, F)^M\).

Next, consider in both networks \( m_r \) defaults. If \( r \) is not involved in any merger, \( r \) would default in both networks and the implications for the proof would be similar. For simplicity, consider that \( r \) is not involved in any merger.

By Assumption 1, \( r \) is the only first step failure in both networks. In the second step, for any \( j \not\in \phi \), if \( \bar{v}_j - l'_j < (\beta'_r + s)\bar{A}_{jr} \) holds in \((C', F')^M\), then \( \bar{v}_j - l_j < (\beta_r + s)\bar{A}_{jr} \) holds in \((C, F)^M\). Similarly, for any \( m_k \), if \( \bar{v}_{mk} - l'_{mk} < (\beta'_r + s)\bar{A}_{mk} \) holds in \((C', F')^M\), then \( \bar{v}_{mk} - l_{mk} < (\beta_r + s)\bar{A}_{mk} \) holds in \((C, F)^M\). Thus, the second step failures in \((C', F')^M\) is a subset of second step failures in \((C, F)^M\), \( \mathcal{N}_2' \subset \mathcal{N}_2 \).

The contagion algorithm implies that the defaults at any step is triggered by the set of defaults up to that step, and hence the same relation holds in the remaining steps of the algorithm as well. As a result, for any \( m_k \), if \( m_k \in \mathcal{N}_t \), then \( m_k \in \mathcal{N}_t \) holds for any \( t \). This completes the proof.

**Proof of Proposition 2:**

Index the mergers in \( M \) based on the distress ranks of the banks involved, and denote a given merger by \( m_k \) if \( \overline{dr}_i = \overline{dr}_j = k \forall (i, j) \in m_k \), and denote the set of mergers \( m_k \) by \( M_k \).

Any merger involving bank \( r \) cannot be a merger of banks with equal distress ranks since \( r \) is the only bank with distress rank 1. Thus, \( r \not\in \phi \) and defaults in \((C, F)^M\).

Suppose that there exists a merger \( m_2 \in M \). We know that \( v_i - l_i - A_{ir} < 0 \) holds for all \( i \in m_2 \). By Lemma 2, \( A_{m2}^M = \sum_{i \in m_2} A_{ir} \). Then, \( \sum_{i \in m_2} (v_i - l_i) - A_{m2}^M = \sum_{i \in m_2} (v_i - l_i) - \sum_{i \in m_2} A_{ir} < 0 \) also holds, which means that \( m_2 \) defaults at step 2 in \((C, F)^M\). This result holds for any \( m_2 \in M_2 \). On the other hand, by Lemma 2, \( A_{jr} = A_{jr}^M \forall j \not\in \phi \), which implies \( v_j - l_j - A_{jr} = v_j - l_j - A_{jr}^M < 0 \) holds for
all \( \{ j : j \notin \phi, \overline{d}r_j = 2 \} \); thus any such bank \( j \) defaults at step 2 in \((C, F)^M\).

Lastly, by some lines of algebra, Lemma 2 implies that any \( \{ j : j \notin \phi, \overline{d}r_j > 2 \} \) remains solvent at step 2 in \((C, F)^M\).

Consider the next step. Suppose that there exists a merger \( m_3 \in M \). So, \( v_i - l_i - A_{ir} - \sum_{k: \overline{d}r_k = 2} A_{ik} < 0 \) holds for any \( i \in m_3 \). By Lemma 2, following the set of failures in step 2 in \((C, F)^M\), a given merger \( m_3 \) defaults if \( \sum_{i \in m_3} (v_i - l_i) - A_{m3r} - \sum_{k \notin \phi} A_{m3k} = 0 \) holds. By Lemma 2, \( A_{m3m2} (\sum_{l \in m2} p_l) = \sum_{k \in m3} (A_{kl} p_l) \) for all \( m2 \in M \). By plugging in this to the default condition for \( m_3 \) above, one can see that merger \( m_3 \) defaults at step 3. Similarly, any bank \( \{ j : j \notin \phi, \overline{d}r_j = 3 \} \) defaults at step 3 in \((C, F)^M\).

By the contagion algorithm, the same result holds in any subsequent step of the algorithm. As a result, any merger \( m_t \in M_t \) defaults at step \( t \) in \((C, F)^M\), and hence there exists no rescue merger in \( M \).

**Proof of Proposition 3:**

First I show that \( v_i = 1 \) holds for all \( i \in N \) in a \( d \)-regular network given that \( p_i = 1 \) for all \( i \in N \).

Consider a \( d \)-regular network \((c, d, F)\) with a given cross-holdings matrix \( C \). Below, I show that \( A = \hat{C}(I - C)^{-1} \) is a row stochastic matrix.

By the Neumann-series representation of the \( A \) matrix, \( A = \hat{C}(I - C)^{-1} = \hat{C} + \sum_{k=0}^{\infty} \hat{C} C^k \).

Consider the cross-holdings matrix \( C \) in a \( d \)-regular network \((c, d, F)\) where the row sum of each row is equal to the column sum of each column, which is equal to \( c \). Next, I show that this regularity condition implies that the row sum of each row in \( C^k \) is equal to \( c \) for \( k \geq 1 \). It holds for \( k = 1 \), immediately. For any \( k > 1 \), I show that result by showing that the row sum of each row in \( C^{k+1} \) is equal to the \( c \) times the row sum of each row in \( C^k \). In matrix representation:

\[
C^{k+1} = \begin{pmatrix}
[C^k]_{11} & \cdots & [C^k]_{1n} \\
\vdots & \ddots & \vdots \\
[C^k]_{n1} & \cdots & [C^k]_{nn}
\end{pmatrix} \begin{pmatrix}
C_{11} & \cdots & C_{1n} \\
\vdots & \ddots & \vdots \\
C_{n1} & \cdots & C_{nn}
\end{pmatrix}.
\]
The row sum of row $i$ of $C^{k+1}$ is equal to $\sum_j [C^k]_{ij} (C_{1j} + \ldots + C_{nj}) = \sum_j [C^k]_{ij} c$. This result holds for any $k$. Thus, the row sum of each row in $C^k$ is equal to $c^k$. Then, the row sum of each row of matrix $(\sum_{k=0}^{\infty} C^k)$ is equal to $c + c^2 + \ldots = 1 - \frac{1}{1-c} = \frac{c}{1-c}$, which implies that the row sum of each row of matrix $A = \hat{C} + \sum_{k=0}^{\infty} \hat{C} C^k$ is equal to $(1-c) + (1-c)\left(\frac{c}{1-c}\right) = 1$ where $\hat{C}_{ii} = 1-c$ for all $i$.

By this result, given that $p_j = 1$ for all $j$, $v_i = \sum_j A_{ij}p_j = 1$ for all $i$.

**Proof of Part i)**

Suppose that $r$ is not involved in any merger. Lemma 1 implies that if there is any bank $\{i : \bar{d}r_i = 2, i \notin \phi\}$, then bank $i$ defaults. Thus, by definition of contagion, each bank $i$ with $\bar{d}r_i = 2$ must be involved in a merger to prevent the contagious failures (all failures but the first failure).

Given that $1-l_i-A_{ir} > 0$ for all $i : \bar{d}r_i \neq 1,2$; then, similar to the argument above, the coalition of set of banks $N \setminus r$ is the most resilient configuration, which means that if the coalition of set of banks $N \setminus r$ can not prevent the contagious failures, then there exists no set $M$ of mergers which can prevent the contagious failures.

By the column stochasticity of the $A$ matrix, the default of bank $r$ reduces the remaining banks’ net worth by $(1-A_{rr})$. In addition, Lemma 2 implies that $A_{Gr} = 1-A_{rr}$ holds. Therefore, $(N-1)(1-l) - A_{Gr} = (N-1)(1-l) - (1-A_{rr}) \geq 0$ must hold.

Lemma 1 implies that for a finite $N$, $A_{ii} > 1-c$ and $1-A_{ii} < c$ for all $i \in N$. Lemma 2 in Elliott et al. (2014) implies that $1-A_{ii}$ is increasing by $c$ for all $c < \frac{1}{2}, i \in N$. Then, there exist $N^*$ and $l^*$ and $c$ such that for $N < N^*$ and $l > l^*$, $(1-A_{rr}(c,d,F)) = (N-1)(1-l) < c < \frac{1}{2}$. Call it $\overline{c}$. If $c > \overline{c}$, then $(N-1)(1-l) < (1-A_{rr}(\overline{c},d,F)) < (1-A_{rr}(c,d,F)) < c$ holds, which means that there exists no merger that can prevent the contagion (all failures but the first failure).

Next, I show that for $d > \overline{d}$, there exists no set of mergers which can prevent all failures but the first failure. For any given $c < \frac{1}{2}$ and $d < \frac{(1-c)c}{(1-l)}$,
any $d$-regular $(c,d,F)$ is contagious. Consider $\bar{c}$. Then, $(\bar{c},d,F)$ is contagious if $d < \frac{(1-c)c}{(1-l)}$. Call that threshold level $\bar{d} = \frac{(1-c)c}{(1-l)}$. Then, for any $d > \bar{d}$, a $(c,d,F)$ being contagious implies that $c > \bar{c}$ since $(1-c)c$ is increasing in $c < \frac{1}{2}$. However for $c > \bar{c}$, I already showed that there exists no merger which can prevent the contagion. Thus, for $d > \bar{d} = \frac{(1-c)c}{(1-l)}$, whenever the financial network is contagious, $c > \bar{c}$ holds, and there exists no merger which can prevent the contagion.

**Proof of Part ii)**

The first failure can be prevented by a merger if the merger can absorb the shock. Then, the grand coalition $G = \{N\}$ is the most resilient merger to prevent the failure of bank $r$ because $\sum_{k \in T} (1 - l_k - sA_{kr}) \leq \sum_{k \in T'} (1 - l_k - sA_{kr})$ for any $T \subset T'$. Thus, if $N(1-l) - sA_{GG} < 0$, then the grand coalition can not prevent the first failure and defaults, which implies that there exists no consortium which can prevent the first failure. By Lemma 2, $A_{GG} = 1$, and the condition can be rewritten as $s > N(1-l)$.

**Proof of Proposition 4:**

Theorem 1.2.5 in Xu (2013) shows that a directed regular network is connected iff it is strongly connected in which there always exists a directed path between any pair of nodes $\{i,j\}$. This implies that a $d$-regular network $(c,d,F)$ defined here is connected iff it is strongly connected. Existence of a directed path between any pair of $\{i,j\}$ in any given connected $(c,d,F)$ implies that following the failure of bank $r$, if contagion occurs, which is guaranteed by $d < \frac{(1-c)c}{(1-l)}$, then all banks in $(c,d,F)$ default. Thus, if $(c,d,F)$ is connected, then there exists no healthy bank in $(c,d,F)$. Hence, if there exists any bank $k$ which is a healthy bank, it must be the case that the given network $(c,d,F)$ is a disconnected network and bank $k$ is in a distinct connected component $K$ s.t. $r \notin K$. Then, By Lemma 1, $A_{ki} = 0$ for any $k \in K$, $i \in R$ where $R$ is the connected component that includes bank $r$. This implies that any healthy bank has net worth of $e_j = 1 - l - A_{ir} = 1 - l$.

Next, consider the set of mergers $M$ as an outcome of the game. Suppose that there exist a healthy bank $k$ and a merger $\{m_k : k \in m_k\}$ such that
denote the network by $(c, d, F)^M$. For $m_k$ to be a rescue merger, there must be a bank $\{i : i \in m_k \cap R\}$. Consider another set of mergers $M' = M \setminus m_k$, an alternative outcome of the game, and denote the network by $(c, d, F)^{M'}$. If $e_k^{M'} = 1 - l > e_k^M$ holds for the healthy bank $k \in m_k$, then the strategy $s'_k = \{k\}$ is always a profitable deviation for $k$, and $M$ can not be an equilibrium set of mergers in that case. So, for $M$ to be an equilibrium outcome, $e_k^M \geq 1 - l$ must hold, and thus $e_m^M \geq 1 - l$ must also hold, where $e_m^M = \alpha_{m_k}^me_{m_k}^M$.

Suppose that $e_{m_k}^M \geq 1 - l$. Then, either there exists at least one more healthy bank in $m_k$, or there exists at least one distressed bank $j$ in $m_k$ such that $j$ is a healthy bank in $(c, d, F)^{M \setminus m_k}$. Otherwise, if all banks in $m_k \setminus k$ are distressed banks in $(c, d, F)^{M \setminus m_k}$, then $v_j^{M \setminus m_k} < l_j$ holds for all $\{j : j \neq k, j \in m_k\}$, which implies that $e_{m_k}^M < 1 - l$. As a result, $M$ can not be an equilibrium outcome.

Next, consider that $k$ is the only healthy bank in $m_k$, but there exists at least one distressed bank $j$ such that $j$ would not default in $(c, d, F)^{M \setminus m_k}$. Denote the set of such banks $\{j : j \in m_k \cap (N_{R \setminus m_k})\}$ by $D$. Then, for any arbitrarily selected sharing rule, $e_j^M = \alpha_{m_k}^je_{m_k}^M \geq 1 - l$ holds if $e_j^M < e_j^{M \setminus m_k} = 1 - l$ holds for some $j \in D$. In this case, $s'_j = \{j\}$ is a profitable deviation for any $\{j : j \in m_k \cap (N_{R \setminus m_k})\}$, and hence $M$ can not be an equilibrium outcome.

Next, consider that there exists more than one healthy bank in $m_k$. In this case, for $e_j^M \geq 1 - l$ to hold, either all banks are healthy banks and reallocate the net worth, which implies that $m_k$ is never a rescue merger, or if $m_k$ is a rescue merger the condition above holds for some $j \in D$ or there exists a healthy bank $i \in m_k$ s.t. $e_i^M < 1 - l$. Then, for any such bank $i$, $s'_i = \{i\}$ is a profitable deviation. As a result, $M$ can not be an equilibrium outcome in this case as well, which completes the proof.

**Proof of Corollary 1:**

Denote the set of banks in the given connected component by $R$. By Lemma 1, $A_{ij} = A_{ji} = 0$ for all $i \in R, j \notin R$. Thus, any bank $j \notin R$ is a healthy bank. Then, Proposition 4 implies that any merger at equilibrium
must be formed among the banks in the set $R$.

Next, consider the mergers in the set $R$. Given that the connected component has size $d + 1$, the connected component is a complete network in which $C_{ij} = \frac{c}{d}$ for all pairs of different nodes $(i, j) \in R$. Thus in a complete component of $R$, $A_{ij} = A_{ik}$ holds for all $j, k$ for $i \neq j, k$. Hence, given that the network is contagious $\overline{dr}_r = 1$ and $\overline{dr}_i = 2$ for all $i \in R \setminus r$. Proposition 2 implies that there exists no merger which can prevent the contagion that is formed among the set of banks $R \setminus r$. Thus, at equilibrium, any merger that can prevent the contagion must involve bank $r$ and some other banks in the set of banks $R \setminus r$. The column stochasticity of the dependency matrix implies that $\sum_{i \in R} A_{ir} = 1$. The merger involving all banks in the set $R$, call it $m_r$, is the most resilient merger in this case since $(1 - l) - sA_{ir} > 0$ for all $i \in R \setminus r$. Consider the merger $m_r = \{R\}$. Then, by Lemma 1, $A^M_{RR} = 1$, and $m_r$ defaults if $(d + 1)(1 - l) - sA^M_{RR} = (d + 1)(1 - l) - s < 0$. As a result, for $s > (d + 1)(1 - l)$, any merger formed among the banks in the set $R$ defaults, and similar to the argument in the proof of Proposition 4, there exists no equilibrium in which the first failure or the contagion is prevented since the banks in the non-contagious region of the network have neither losses nor gains from the costly rescues.

**Proof of Proposition 5:**

**Proof of Part i)**

First, I show that for $s \leq s^*$, there exists an equilibrium such that the first failure is prevented in $(c, d, F)^M$. Consider any set of strategies $s^*$ restricted by the assumption $S_i = \{T \mid T \subset \{i, (N \setminus r), N\}\}$. Then, the outcome of the game is either $M = \emptyset$ or $M = \{(N \setminus r), r\}$ or $M = \{N\}$. For $\{N\}$ to be the outcome of the game, $\{N\} \subset s_i$ must hold for all $i$. Suppose that $\{N\} \subset s_i$ holds for all $i$. Next, I show that for any such set of strategies, and for the shock level such that $s \leq s^*$, there exists no coalitional deviation.

For any such set of strategies, the outcome of the game is $M = \{N\}$, and the payoffs in $(c, d, F)^M$ are equal to $e_i^M = \alpha_i^M e_{\{N\}}^M = \alpha_i^M [N(1 - l) - sA_G]$ for all $i \in N \setminus r$ where $G$ is the grand coalition of all banks in the set $N$. By Lemma
2, $A_{GG} = 1$, and $e_i^M = \alpha_i^M e_{\{N\}}^M = \alpha_i^M [N(1 - l) - s]$ for all $i \in N \setminus r$. On the other hand, if the coalition of $T = (N \setminus r)$ deviates such that $s'_i = \{N \setminus r\}$ for all $i \in N \setminus r$, then the outcome of the game is $M' = \{(N \setminus r), r\}$. In that case, by Lemma 2, $e_i^{M'} = \alpha_i^{M'} [(N - 1)(1 - l) - \sum_{i \in N \setminus r} A_{ir}] = \alpha_i^{M'} [(N - 1)(1 - l) - (1 - A_{rr})]$ for all $i \in N \setminus r$.

Then, for any given sharing rule $\alpha$ such that $\alpha_{\{N\}} = 0$, for $e_i^{M'} < e_i^M$ to hold, it must be the case that the total welfare of the members of the coalition would be lower after the coalitional deviation that results in the formation of $M' = \{(N \setminus r), r\}$. This is true if $[N(1 - l) - s] < [(N - 1)(1 - l) - (1 - A_{rr})]$. Thus, for $[N(1 - l) - s] \geq [(N - 1)(1 - l) - (1 - A_{rr})]$, there always exists a bank $i \in N \setminus r$ s.t. $e_i^{M'} < e_i^M$. Therefore, there exists no such coalitional deviation. On the other hand, a deviation of $s_i = \{i\}$ is never profitable for any bank $i$, because otherwise no rescue merger is formed and all banks default in the given contagious, connected $d$-regular network $(c, d, F)$. Thus, there also exists no such coalitional deviation. These together imply that any set of strategies $s^*$ s.t. $\{s_i : \{N\} \subset s_i\}$ for all $i$, is an equilibrium set of strategies, and $(c, d, F)^{(N)}$ is the equilibrium network. The threshold level of the shock can be rewritten as $\overline{s} = (1 - A_{rr}) + (1 - l)$. Thus, for $s \leq \overline{s}$, $(c, d, F)^{(N)}$ is an equilibrium network.

**Proof of Part ii)**

Next, consider that $s > \overline{s} = (1 - A_{rr}) + (1 - l)$. In this case, the set of strategies $s^*$ s.t. $\{s_i : \{N \setminus r\} \subset s_i\}$ for all $i$, is an equilibrium set of strategies and $(c, d, F)^{(N \setminus r), r}$ is the equilibrium network. The reason is similar to the argument above. For any given sharing rule $\alpha$ such that $\alpha_{\{N\}} = \alpha_{(N \setminus r), r} = 0$, if $[N(1 - l) - s] < [(N - 1)(1 - l) - (1 - A_{rr})]$, then there always exists a bank $i \in N \setminus r$ s.t. $e_i^{M'} < e_i^M$ for $M' = \{N\}$. Thus, there exists no such coalitional deviation.

Similarly, if there exists any bank $i$ s.t. $s_i = \{i\}$, then the outcome of the game is $M = \emptyset$, which results in default of all banks in $(c, d, F)^M$. Thus, for $s > \overline{s}$, there exists an equilibrium such that $(c, d, F)^{(N \setminus r), r}$ is the equilibrium network.
PROOFS OF PROPOSITION 6 AND 7:

First, I show that \( \{A_{ii}\}^d = \{A_{jj}\}^d \) holds for any given \( d \)-ring lattice \((c, dR, F)\).

The \( C \) matrix of a given \((c, dR, F)\) is a circulant matrix, where each row vector is rotated one element to the right relative to the preceding row vector.

By Theorem 3.2.4 of Davis (2012), if \( A \) and \( B \) are circulant matrices of order \( n \) and \( \alpha_k \) scalars, then \( \alpha_1 A + \alpha_2 B \) and \( AB \) are also circulant matrices. This theorem implies that \( I - C \) is also a circulant matrix, since \( I \) is a circulant matrix as well. Theorem 3.2.4 of Davis (2012) shows that if \( A \) is a non-singular circulant matrix, then \( A^{-1} \) is also a circulant matrix. Given that \( (I - C) \) is invertible, \((I - C)^{-1}\) is also a circulant matrix. Finally, \( \hat{C} (I - C)^{-1} \) is a circulant matrix since \( \hat{C} \) and \((I - C)^{-1}\) are circulant matrices of order \( n \). This implies that \( \{A_{ii}\}^d = \{A_{jj}\}^d \) holds for any given \( d \) where \( \{A_{ii}\}^d \) is the \( A_{ii} \) of bank \( i \) in \((c, d, f)\) with diversification level \( d \).

Next, I show that for \( c \leq c \leq \bar{c} \), \( d \leq \bar{d} < \frac{(1-c)\bar{c}}{1-c} \), \( d \)-ring lattice \((c, dR, F)\) is optimal, and I also show that any non-contagious \( d \)-ring lattice \((c, dR, F)\) is always dominated by a contagious \( d \)-ring lattice \((c, dR, F)\), which is the Proof of Proposition 7.

If a network satisfies the following properties, then it is an optimal network:

i) for \( s = s_s \), there exists a coalition-proof Nash equilibrium in which the first failure is prevented by a rescue consortium,

ii) for \( s = s_L \), there exists a coalition-proof Nash equilibrium in which the contagion (all failures but the first failure) is prevented by a rescue consortium, or there exists no contagion.

Any network satisfying the properties above weakly dominates any other network because the social welfare in such a network is equal to \( W(c, d, F)^M = q(\sum_{i \in N^M} p_i) - s + (1 - q)(\sum_{r \in N^M \setminus F} p_r) \), which is the maximum possible value for \( W(c, d, F)^M \) in any network for the given shock and external liabilities for the given set of banks.

For a \( \psi(C, F) \in \Omega \) to have the properties above; by Proposition 5, the condition for preventing the first failure rather than the contagion, for a con-
tagious network is such that:

\[ [N(1-l) - s] \geq [(N-1)(1-l) - (1 - A_{rr}(C, F))] \]  

(7)

must hold for \( s = s_S \) for all \( r \in N \), and by Proposition 3, the condition for rescuable contagion, for a contagious network:

\[ [(N-1)(1-l) - (1 - A_{rr}(C, F))] \geq 0 \]  

(8)

must hold for \( s = s_L \) for all \( r \in N \). In any non-contagious network, equation (8) strictly holds.

In a \( d \)-ring lattice \((c, d^R, F)\), equation (7) holds for any value for \( s_s \leq N(1-l) \) if \([(N-1)(1-l) = (1 - A_{rr}(c, d^R, F))]\). By Proposition 3, there exist \( N^* \), \( l^* \) and \( c \) such that for \( N < N^* \) and \( l > l^* \), \((1 - A_{rr}(c, d^R, F)) = (N-1)(1-l) < c < \frac{1}{2} \). As shown in the proof of Proposition 3, the integration level \( c \) is bounded from below by \( c > (N-1)(1-l) \) such that \((N-1)(1-l) = (1 - A_{rr}(\zeta, d^R, F)) < \zeta \), and it is bounded from above by \( \bar{c} \) s.t. \((N-1)(1-l) = (1 - A_{rr}(\bar{c}, d^R, F)) < \bar{c} \). So, for \( N(1-l) < N^*(1-l) \) \( < \frac{1}{2} \), there always exists a CPNE in which the first failure is prevented in a contagious \( d \)-ring lattice \((c, d^R, F)\) such that \( d^R < \frac{c(1-l)}{1-l} \). On the other hand, if the exogenously given \( N \) and \( l \) are such that \( N \geq N^* \) and \( l \leq l^* \), then there exists no integration level \( c \) satisfying \([ (N-1)(1-l) = (1 - A_{rr}(c, d, F))] \) in any network \( \psi(C, F) \in \Omega \).

Thus, for small enough \( \epsilon \), the first failure always occurs in any network \( \psi(C, F) \in \Omega \) if \( N \geq N^* \) and \( l \leq l^* \).

If the network is non-contagious, then \([ (N-1)(1-l) - (1 - A_{rr})] \) is always positive and for small enough \( \epsilon \), \([N(1-l) - s_s] = [N(1-l) - (N(1-l) - \epsilon)] < [(N-1)(1-l) - (1 - A_{rr})]\) holds. Thus, preventing the first failure becomes not profitable in a non-contagious \( d \)-ring lattice \((c, d^R, F)\).

Next, consider the large shock, \( s_L \). The first failure can not be prevented in any network when \( s = s_L \), independently from whether the network is contagious. The desired condition in this case is given by Equation (8).

If the equation (8) strictly holds, then the condition in equation 7 is violated for small enough \( \epsilon \). Then, we are left with the condition that \([ (N-1)(1-l) =\)
\((1 - A_{rr}(c, d, F))\). Thus, for \(N < N^*\) and \(l > l^*\), \(\xi \leq c \leq \bar{c}\) is a preferred level of integration in this case, since it guarantees that the contagion is prevented when \(s = s_L\), and does not violate the condition in equation 7. On the other hand, the contagion is always preventable in any network \(\psi(C, F) \in \Omega\) if \(N \geq N^*\) and \(l \leq l^*\).

Combining the results together, any contagious \(d\)-ring lattice \((c, d, F)\) with \(\xi \leq c \leq \bar{c}\) and \(d^R < \frac{\xi(1-c)}{1-l}\) is an optimal network. In such a \(d\)-ring lattice \((c, d^R, F)\), \(A_{ii} = A_{jj}\) for all \(i, j \in N\). So, the results hold for an arbitrarily selected \(r \in N\).

**PROOF OF PROPOSITION 8:**

For such a network \((c, d, F)\), \(\lfloor N(1 - l) - s \rfloor < 0\) and \(\lfloor (N - 1)(1 - l) - (1 - A_{rr}(C, F)) \rfloor < 0\). The required government transfer to prevent the first failure is at least \(t_N = s - N(1 - l)\). The social welfare in this case is equal to \(W(c, d, F)^{M^*_\psi} = (\sum_{i \in N^M} p_i) - s - (s - N(1 - l))\), if the grand coalition with the government transfer is formed and the first failure is prevented.

On the other hand, if the government does not assist the rescue of the shocked bank, but assists in the rescue of the contagious failures, then the required government transfer is equal to \(t_{\{N \backslash r\}} = [(1 - A_{rr}(C, F)) - (N - 1)(1 - l)]\). In that case, the social welfare is equal to \(W(c, d, F)^{M^*_\psi} = (\sum_{i \in N^M} p_i) - s - [(1 - A_{rr}(C, F)) - (N - 1)(1 - l)]\), if the grand coalition excluding bank \(r\) is formed with the government transfer, and the contagious failures are prevented.

Therefore, for \(\sum_{i \in N^M} p_i - s - [(1 - A_{rr}(C, F)) - (N - 1)(1 - l)] \leq \sum_{i \in N^M} p_i - s - (s - N(1 - l))\), the government prefers to assist in the rescue of the shocked bank. Given that with no intervention all banks would fall and the shareholders payoff would be equal to zero, then there exists no coalitional deviation as shown in the proof of Proposition 5, and the shocked bank would be rescued at equilibrium with the government transfer.

The condition for the shock can be rewritten as \(s \leq s^* = [(1 - A_{rr}(C, F)) + (1 - l)]\). For given vector of external liabilities \(l\), \(s^*\) is increasing in \(1 - A_{rr}(C, F)\), which is increasing the integration level \(c\), by Lemma 2 in Elliott et al. (2014).
For, $s > s^* = [(1 - A_{rr}(C, F)) + (1 - l)]$, the government prefers to assist in the rescue of the rest of the system, which completes the proof.

**PROOF OF PROPOSITION 9:**

Proposition 6 implies that for $s = s_S$, the first failure is prevented in an optimal network. Thus, for $s = s_S$, there is no need for government assistance. Next, I show that there exists an upper bound $\overline{s} = \frac{1 + N(1 - l)}{2}$ such that the first failure is prevented by a government-assisted rescue merger if the large shock hits ($s = s_L$).

For $s = s_L$, the social welfare with no government assistance in an optimal network is equal to $W(c, d, F)^{M^*_\psi} = (\sum_{i \in N} p_i) - p_r$ where $M^*_\psi$ is the set of mergers in the best-case equilibrium in $\psi$, in which the contagious failures are prevented. On the other hand, if the government assists merger $m_k$ which prevents the first failure, then the social welfare is equal to $W(c, d, F)^{M^{**}\psi} = (\sum_{i \in N} p_i) - s_L - t_{m_k}$ where $M^{**}\psi$ is the set of mergers in the best-case equilibrium in $\psi$, and $m_k \in M^{**}\psi$ is the government-assisted merger which prevents the first failure.

Thus, the government assists the rescue of first failure if $(\sum_{i \in N} p_i) - s_L - t_{m_k} \geq (\sum_{i \in N} p_i) - p_r$, which can be rewritten as $N - s_L - t_{m_k} \geq N - 1$.

For $s_L > N(1 - l)$, the required amount of the government transfer to assist $m_k = \{N\}$ is equal to $t_{\{N\}} = s_L - N(1 - l)$. Proposition 7 implies that the announcement of the transfer of $t_{\{N\}} = s_L - N(1 - l)$ results in the prevention of the first failure at equilibrium in an optimal network.

So, the government assists $m_k = \{N\}$ iff $N - s_L - [s_L - N(1 - l)] \geq N - 1$, which can be rewritten as $s_L \leq \frac{1 + N(1 - l)}{2}$.  

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The Linearities in Interbank Contracts.—Figure 5 illustrates how a decline in the proprietary asset return of bank $i$ affects the interbank liabilities and the net worth of bank $i$ and bank $j$.

**Figure 5:** Interdependencies in the balance sheets

Figure 5 shows the increase in leverage ratios following a reduction in a proprietary asset return, where the external liabilities ($l_i$) of each bank is fixed
and given exogenously. Hence, a significant drop in proprietary asset returns might cause bankruptcies. The figure above illustrates the simplified case with two banks, whereas the model captures the complex environments in which there exist more than two banks.