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Working Paper #19-02

September 2019

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Product Rankings, Customer Search and Fake Orders**

Chen Jin

National University of Singapore, School of Computing

Luyi Yang

Johns Hopkins University, Carey Business School

Kartik Hosanagar

University of Pennsylvania, The Wharton School

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To Brush or Not to Brush: Product Rankings, Customer Search, and Fake Orders*

Chen Jin

National University of Singapore, School of Computing, Department of Information Systems and Analytics, disjinc@nus.edu.sg

Luyi Yang

Johns Hopkins University, Carey Business School, luyi.yang@jhu.edu

Kartik Hosanagar

University of Pennsylvania, The Wharton School, kartikh@wharton.upenn.edu

“Brushing”—the practice of online merchants placing fake orders of their own products to artificially inflate sales on e-commerce platforms—has recently received widespread public attention. On the one hand, brushing enables merchants to boost their rankings in search results, because products with higher sales volume are often ranked higher. On the other hand, rankings matter because search frictions faced by customers narrow their attention to only the few products that show up at the top. Thus, fake orders from brushing may affect customer choice. We build a stylized model to understand merchants’ strategic brushing behavior and its welfare implications. We consider two competing merchants selling substitutable products (one of high quality, the other of low quality) in an evolutionary sales-based ranking system that assigns a higher ranking to a product with higher sales. In principle, such an adaptive system improves customer welfare relative to a case in which products are randomly ranked, but it also triggers brushing as an unintended consequence. Since the high-quality merchant receives a favorable bias in the sales-based ranking, he mainly has a defensive brushing incentive, whereas the low-quality merchant mostly has an offensive brushing incentive. As a result, brushing is a double-edged sword for customers. It may lead customer welfare to be even lower than what it would be in a random-ranking system, but in some other cases, it can surprisingly improve customer welfare. If brushing is more difficult for merchants (e.g., due to tougher regulations), it may make customers worse off as it attenuates brushing by the high-quality merchant but induces the low-quality one to brush more aggressively. If search is easier for customers (e.g., due to improved search technologies), it can actually hurt them as it may disproportionately discourage the high-quality merchant from brushing.

Key words: search, rankings, brushing, fake, customer welfare

* We are grateful to the NET Institute (www.NETInst.org) for funding this research.

Good wine needs no bush.

—William Shakespeare

1. Introduction

“Brushing”—online merchants placing fake orders of their own products to pad their sales figures—has been an increasingly pervasive practice witnessed on major e-commerce platforms such as Alibaba’s Taobao (WSJ 2015b) and Tmall sites (FT 2016), JD.com (Forbes 2015) and Amazon (CBS 2018). Here is how brushing works: merchants reach out to professional brushers, who place orders and make payments using the money they receive from the merchants; the merchants ship out empty parcels or boxes of worthless items; the brushers get compensated and the fake transactions increase the sales volume of the merchants, enabling their products to rank higher in search results (WSJ 2015b, FT 2016). According to an estimate by Alibaba’s Vice President Yu Weimin, 1.2 million merchants on Taobao—or about 17% of all the vendors—had faked 500 million transactions worth 10 billion RMB in the year of 2013 alone; he further said those were “only the tip of the iceberg,” and his conservative estimate put the number of brushers in the tens of thousands (WSJ 2015b).

What drives brushing? According to a Taobao merchant, “without fake transactions, your product will end up at the very back of the search results, and people will never be able to find it;” this is echoed by a Taobao merchant who admits to brushing in the past: “how do you get yourself noticed by customers in the sea of ... products if you don’t have a single sale?” (WSJ 2015b). “The difference between being at the top of a page of results and buried at the bottom is night and day; brushing is a very tempting shortcut,” says an industry observer (FT 2016).

These quotes reveal the driving forces behind brushing. Customers face *search frictions*, and thus often only consider prominent products that rank high in search results. At the same time, platforms’ *ranking algorithm* tends to push products with higher sales volume to the top. These two factors combined create a feedback loop between rankings and sales: a higher ranking drives more sales (due to search frictions), which in turn, lead to a higher ranking (due to the sales-based ranking algorithm). Therefore, vendors are naturally under the pressure to rack up fictitious sales to improve their rankings and attract real customers.

Platforms often claim they do not condone brushing. To combat such a practice, they claim that they deploy sophisticated analytics tools based on data mining and machine learning to detect and remove fake transactions (Bloomberg 2017, WSJ 2015a), albeit with limited success (FT 2016, WSJ 2015a). A study by Xu et al. (2017) finds only a small proportion of the sellers involved in brushing are detected and materially penalized. The challenges are manifold. The sheer volume of transactions on these online marketplaces is one impediment (WSJ 2015a). Also, real shipping and

delivery take place despite fake transactions (WSJ 2015b). Moreover, brushers often mimic real shoppers’ browsing and clicking behavior to evade detection (NPR 2018). While platforms’ effort to crack down on brushing cannot easily eliminate brushing, it obviously makes brushing more difficult. An implicit justification for platforms’ intervention is that if brushing is more difficult, sellers brush less and customers are better off. Such a customer-centric view is crucial for platforms’ long-term success, but is this argument logically sound?

To that end, our paper aims to understand online sellers’ strategic brushing behavior and its implications for customer welfare. We build a stylized model to address the following research questions: (1) How do sellers strategically brush in this competitive environment? (2) What is the impact of brushing on customer welfare? (3) Given platforms’ effort to combat brushing and the difficulty involved along the way, is the effort truly justifiable? Specifically, will customers necessarily benefit if brushing is made more difficult? (4) Given that customers’ search frictions are responsible for the occurrence of brushing, will a lower search cost necessarily dampen brushing incentives and improve customer welfare? (5) Given that the sales-based ranking algorithm creates an incentive for brushing, what is the rationale for such an algorithm and what happens if products are randomly ranked irrespective of their past sales?

Our model considers two merchants selling substitutable products on an e-commerce platform. One merchant sells a high-quality product and the other, a low-quality one. Customers arrive sequentially and each customer is interested in buying one product. Upon arrival, each customer is presented with a ranking (i.e., an ordered list) of the two products. She knows one is of high quality and the other, low quality, but does not know which is which and whether either one fits. To resolve such uncertainty, she conducts a sequential search. A low-quality product that fits is less valuable than a high-quality product that does, but more valuable than a product that does not fit, which is of equal worth regardless of its type. As the two products look identical *ex ante*, she starts her search from the top, and incurs a search cost if she also searches the second product (the one at the bottom). In the absence of search frictions, rankings do not matter, and a customer is always more likely to buy a high-quality product. Nevertheless, in the presence search frictions, rankings make a difference, and a customer may be more likely to buy a low-quality product if it is placed at the top.

We consider two ranking systems: (1) random ranking, which presents each customer with a random permutation of the two products; (2) sales-based ranking, whereby the product with higher sales volume is ranked above the other. The random ranking is an open-loop scheme, whereas the sales-based ranking is an evolutionary procedure. We model the evolution of the two products’ sales difference in the sales-based ranking system as a generalized random walk. We show that the sales-based ranking system is more likely to push high-quality products to the top than the

random ranking system, and therefore, it (weakly) improves customer welfare. This result provides a rationale for why the sales-based ranking is commonly used in practice.

However, this result is established with the caveat that sellers are not strategic. In reality, the strength of the sales-based ranking system also becomes its own weakness: sellers may want to game such a ranking system by placing fake orders (they apparently have no incentive to do so under random ranking). To capture sellers' strategic reactions, we consider a brushing game between the two sellers. Two incentives underlie brushing: the incentive to get a head start (especially if the opponent does not brush), termed the *offensive incentive*; and the incentive to avoid lagging behind (especially if the opponent brushes), termed the *defensive incentive*. Since the high-quality seller receives a favorable bias in the sales-based ranking system, he employs brushing mostly to defend his prerogative, whereas the low-quality seller engages in brushing as a means to challenge the high-quality seller's pre-existing advantage.

We characterize the brushing equilibrium. When the brushing cost is sufficiently high, neither seller brushes because it would be too costly to offend; when the brushing cost is sufficiently low, both sellers engage in full brushing (fake-ordering the maximum quantity) to defend themselves. When the brushing cost is intermediate, both sellers engage in partial brushing, but at different intensities. When the brushing cost is intermediately high, we find the high-quality seller brushes less than the low-quality seller. In this case, brushing would be mostly due to the offensive incentive, which the high-quality seller lacks; he finds brushing too costly and thus does not brush much. Consequently, the low-quality seller seizes the opportunity and brushes intensely to tip over the balance. When the brushing cost is intermediately low, we find the high-quality seller brushes more than the low-quality seller. In this case, brushing would be mostly due to the defensive incentive, driven by which, the high-quality seller brushes aggressively because brushing is cheap, and the low-quality seller, in turn, capitulates and brushes at a low intensity. Consequently, as the brushing cost decreases, the high-quality seller, as expected, brushes more, but the low-quality seller, on the contrary, may brush less as a strategic response to the high-quality seller's behavior.

The presence of brushing may completely negate the welfare advantage the sales-based ranking system has over random ranking. In fact, customer welfare can even be lower in the former system when both the search cost and brushing cost are intermediately high. However, under some other circumstances (specifically, intermediately low brushing cost and intermediately high search cost), the presence of brushing can surprisingly benefit customers, further enhancing customer welfare in the sales-based ranking, as the high-quality seller brushes substantially more than the low-quality one. Contrary to conventional wisdom, if brushing is made more costly (presumably due to harsher policing by the platform), customer welfare may surprisingly decline, because, as we explained earlier, the high-quality seller would be discouraged from brushing while the low-quality

seller would brush more. From a managerial standpoint, these results imply that an escalated combat against brushing need not help customers. Instead of trying to weed out the brushing practice altogether (which would be very difficult anyway), platforms should first investigate which sellers are brushing more aggressively. If brushing is more rampant among high-quality sellers, then platforms can adopt a more *laissez-faire* policy towards brushing.

Another potential solution to addressing brushing is to reduce customer search frictions (e.g. improved search technologies or more user-friendly interfaces). While reducing search frictions generally dampens both sellers’ brushing effort, it can sometimes discourage the high-quality seller more than it does the low-quality seller. As a result, customer welfare can decline. When the search cost is sufficiently high, customers do not search and purchase the top-ranked product regardless of its type, and therefore, the two sellers adopt the same brushing strategy that offsets each other. Nevertheless, when the search cost is lower (at an intermediately-high level), customers can exercise some discretion in search (and are thus biased toward the high-quality product) but is still subject to the ranking effect; this produces asymmetric brushing incentives for the two sellers, and when the brushing cost is intermediately high, the low-quality seller brushes more aggressively than the high-quality seller, which can lead to lower customer welfare than if the search cost is high. From a managerial standpoint, this implies that when deciding whether reducing search frictions is a worthy investment, platforms should weigh the benefits from lower search frictions against the possibility of increasing the relative brushing intensity by low-quality sellers.

2. Literature Review

One key premise of our paper is that product rankings matter and specifically, a higher ranking attracts more traffic. The empirical literature on product rankings lends support to such a ranking effect. Agarwal et al. (2011) show in the context of sponsored advertising that an ad being placed at a higher position enjoys a higher click-through rate (CTR). Using hotel-search data, Ghose et al. (2014) document a significant interplay between product ratings and search engine rankings. Relatedly, Jeziorski and Moorthy (2018) demonstrate the ranking effect is more sizable for advertisers with less brand prominence. Ursu (2018) quantifies the ranking effect in an Expedia dataset by structurally estimating a sequential search model in the spirit of Weitzman (1979).

To that end, our paper builds on the extensive consumer search literature. Particularly relevant to our paper are those that theoretically investigate the impact of firm prominence on search markets (Armstrong et al. 2009, Armstrong and Zhou 2011). This stream of literature operates under the assumption that a firm that is more prominently placed will be sampled earlier by customers and firms can strategically choose to be more prominent—namely, influencing the order in which they are being considered—at a cost. The practice of brushing examined in our paper is also a costly means to gaining prominence, and is similar to bidding for top positions in online ad markets.

As such, our paper is closely related to the literature on position auctions in sponsored search advertising whereby advertisers bid for ad placement. Edelman et al. (2007), Varian (2007), Katona and Sarvary (2010), Abhishek and Hosanagar (2013), Ye et al. (2015) assume exogenous CTRs that are decreasing from top to bottom, without modeling how CTRs are generated by customer search behavior. Other papers integrate consumer search into position auctions as micro-foundations for the differences in CTRs across ad positions (e.g., Athey and Ellison 2011, Jerath et al. 2011, Chen and He 2011, Chu et al. 2019). Our work follows this line of research and explicitly model customer search.

In all the above papers on firm prominence and position auctions, the ordering of firms is static, whereas the sales-based ranking we focus is dynamic and evolves organically as a result of customer search and purchase. As such, it can be seen as a combination of organic listing (because of the ranking evolution) and sponsored listing (because of brushing). Thus, our paper is related to Katona and Sarvary (2010) and Xu et al. (2012), who study the role of organic listing in sponsored search advertising, although they do not consider the ranking- and sales-dynamics as we do.

The fraudulent nature of brushing connects our paper with the literature on click fraud in search advertising (Wilbur and Zhu 2009, Chen et al. 2015), which focuses on a very different problem setting. The typical motivation for committing click fraud, namely, deceptively clicking on search ads, is either to increase the third-party search engine’s revenue or to deplete a competitor advertiser’s budget. By contrast, the motivation for brushing is to manipulate the rankings and generate more sales of one’s own product. In a similar vein, our paper complements the literature on fake reviews (Mayzlin 2006, Mayzlin et al. 2014, Anderson and Simester 2014, Luca and Zervas 2016, Lappas et al. 2016) and false advertising (Corts 2013, 2014, Piccolo et al. 2015, Zinman and Zitzewitz 2016, Rao and Wang 2017, Rhodes and Wilson 2018, Piccolo et al. 2018). Fake reviews and false advertising are meant to influence customers’ evaluation of a product, whereas fake orders in our context are meant to affect whether customers consider a product, but not their actual product valuation.

From a modeling perspective, Fleder and Hosanagar (2009) are particularly relevant to our paper. They consider a recommender system in a two-product setting that always recommends a product that sells more in the past. In light of the recommendation, each customer chooses one of the two products on the basis of exogenously imposed decision rules rather than utility maximization as in our model. They build a generalized random-walk model to capture the evolution of the two products’ sales difference. Our model of the sales-based ranking system bears a resemblance to, but is appreciably different from theirs. Our model is more intricate in that we explicitly model rankings; our analysis is more involved as we show how the evolution of the random walk is susceptible to sellers’ brushing behavior.

To the best of our knowledge, while ours is the first analytical paper to formally model brushing, two papers have empirically investigated this practice. Xu et al. (2017) measure the scale of brushing on Taobao through web crawling and identify more than 11,000 sellers as faking transactions in a two-month period. They find that brushing can substantially increase an online seller’s reputation. Using a rich dataset that consists of more than 300,000 products listed on a major e-commerce platform in a three-month period, Wang et al. (2018) find that brushing generates more traffic in the short run, but has a negative effect on product performance in the long run. The focus of both papers is on measuring the impact of brushing on the sellers (which is still inconclusive as the findings of the two papers are somewhat contradictory) given the variations in sellers’ endogenous brushing behavior, but they leave open the fundamental question of why sellers differ how they brush in the first place. We complement these empirical studies by providing a theoretical understanding of sellers’ brushing behavior and its welfare implications.

3. Model Setup and Preliminary Analysis

We consider two competing sellers, H and L , on an e-commerce platform selling products H and L , respectively. Customers arrive in discrete time periods over an infinite horizon. In each period, one customer arrives and is interested in purchasing one product from one of the two sellers. We assume that the platform has no knowledge of the customer arrival process. Each customer’s net valuation for product $i \in \{H, L\}$ is v_i if it fits her and v_0 , otherwise. A product fits a customer with probability $\gamma \in (0, 1)$. We assume $v_H > v_L > v_0 > 0$. That is, if both products fit, product H is preferred (e.g., product H includes free shipping but product L does not; see Jerath et al. 2011); but a fitting product L is preferred to an unfit product H (e.g., product H is not of the desirable color, but product L is). For expository convenience, we refer to seller/product H (resp. L) as the high- (resp. low-) quality seller/product. In essence, our model captures a setting in which products are both vertically and horizontally differentiated.

After an arriving customer types in her search query, she is presented with an ordered list (ranking) of the two products. Each customer knows that one is H and the other, L , but does not know which is which or whether either one fits. The platform does not have such knowledge, either. To resolve this uncertainty, each customer conducts a sequential search, e.g., clicking on the product links one by one to find out more about the products (e.g., shipping options, color, texture, etc). After a customer searches a product, she identifies her net valuation for it (including both whether it is H or L and whether it fits). Since the two products look identical before a customer searches, we assume she starts her search from the top. This behavioral assumption is in line with the empirical literature on how online consumers process information from a serial list (Hoque and Lohse 1999). In §6.1, we further consider an extension that allows some customers not

to search from the top and demonstrate the robustness of our insights. After the net valuation of the top-ranked product is revealed, each customer decides whether to purchase it or search the second one at the bottom to maximize her expected utility. If she searches, she pays search cost c and discovers her net valuation of the second product, at which point, she knows her net valuation of both products and purchases the one with a higher net valuation. If there is a tie (i.e., neither fits), she randomly purchases a product with equal probability ($1/2$). Exactly one product is purchased at the end of each period. Table 1 summarizes the main notation used in the paper.

The way we model customer search captures the ranking effect whereby a higher ranking attracts more traffic. Specifically, in our model, the product placed at the top is searched and considered by every customer, whereas the product at the second position may not be. For ease of exposition, we restrict attention to the case $v_H + v_0 \geq 2v_L$ (Assumption 1) in the main body of the paper. We present our analysis for the other case $v_H + v_0 < 2v_L$ in Appendix B, and the results only bear cosmetic differences.

Table 1 **Glossary of Main Notation**

Symbol	Definition
c	Search cost
γ	Fit probability
v_H, v_L	Net valuation of a fitting product H (resp. L)
v_0	Net valuation of an unfit product
δ	Discount factor
R	Reward from selling one unit
c_B	Cost of brushing one unit
Q	Maximum brushing quantity
$\mathcal{S}$	State space of the random walk, $\mathcal{S} = (\mathbb{Z} \setminus \{0\}) \cup \{\bar{0}, \underline{0}\}$
p_1, p_2	Purchase probability of product H when it is ranked 1st (resp. 2nd)
U_H, U_L	Customer's expected utility upon arrival if product H (resp. L) is top-ranked
β_H, β_L	Brushing probability of seller H (resp. L)
π_H, π_L	Total expected discounted profit of seller H (resp. L)
W_R, W_S, W_B	Customer welfare under random ranking, sales-based ranking, and brushing, respectively
$P(s)$	Probability that the random walk drifts to $+\infty$ when the initial state is s
$V_H(s), V_L(s)$	Total expected discounted reward of seller H (resp. L) when the initial state is s
M_H, M_L	Marginal gain in reward from brushing for seller H (resp. L) when neither seller brushes
M'_H, M'_L	Marginal loss in reward from not brushing for seller H (resp. L) when both sellers brush
$\bar{C}, \underline{C}, \bar{C}', \underline{C}'$	Thresholds on brushing cost c_B
$\bar{\gamma}$	Threshold on the fit probability γ
α	Fraction of informed customers with zero search cost

ASSUMPTION 1. $v_H + v_0 \geq 2v_L$.

Let p_1 (resp. p_2) be the probability of a customer choosing product H when it is ranked first (resp. second). Let U_i denote the expected utility for an arriving customer (before search) if product i is ranked first (at the top), where $i \in \{L, H\}$. Proposition 1 below characterizes customer's optimal search strategies and the associated search outcomes.

PROPOSITION 1. *Customers' search strategies are summarized in Table 2.*

Table 2 Customers' Search Strategies			
Top-Ranked Product	Search Cost c	Realized Valuation	Strategy
H	$c \geq \gamma(v_L - v_0)$	v_0 or v_H	Not Search
	$c < \gamma(v_L - v_0)$	v_0 v_H	Search Not Search
L	$c \geq \gamma(v_H - v_0)$	v_0 or v_L	Not Search
	$\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$	v_0 v_L	Search Not Search
	$c < \gamma(v_H - v_L)$	v_0 or v_L	Search

Customers' purchase probabilities p_1, p_2 , and expected utilities U_H, U_L are summarized in Table 3.

Table 3 Customers' Purchase Probabilities and Expected Utilities			
Case	Search Cost c	Purchase Probabilities p_1, p_2	Expected Utilities U_H, U_L
1.1	$c \geq \gamma(v_H - v_0)$	$p_1 = 1$ $p_2 = 0$	$U_H = \gamma v_H + (1 - \gamma)v_0$ $U_L = \gamma v_L + (1 - \gamma)v_0$
1.2	$\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$	$p_1 = 1$ $p_2 = (1 - \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)v_0$ $U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]$
1.3	$\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$	$p_1 = 1$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)v_0$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - c$
1.4	$c < \gamma(v_L - v_0)$	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c]$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - c$

The proofs of Proposition 1 and the rest of the main results are relegated to Appendix A. Table 3 divides the search cost into four cases. In Case 1.1, the search cost is so high that customers always purchase the top-ranked product without search. Hence, the purchase probabilities p_1, p_2 are product-independent. In Case 1.4, the search cost is so low that customers always search to the extent that they find the best product that fits their need. Hence, the purchase probabilities are ranking-independent. In Cases 1.2 and 1.3, the purchase probabilities are both ranking- and product-dependent. In Case 1.3, the search cost is intermediately low; customers search to a considerable degree, and therefore, the product effect outweighs the ranking effect in that customers are still more likely to buy product H even when it is buried at the bottom (i.e., $p_2 > 1/2$). In Case 1.2, the search cost is intermediately high, customers perform limited search, and therefore, the ranking effect overshadows the product effect in that customers are less likely to buy product H when it is placed at the bottom (i.e., $p_2 < 1/2$).

To sharpen our understanding of customer search in this problem and the impact of product rankings, Corollary 1 distills some key properties from Proposition 1.¹

¹ When we say "increasing" or "decreasing," we mean it in a weak sense.

COROLLARY 1. *Customers are:*

- (i) *more likely to purchase a product if it is top-ranked than if it is not, i.e., $p_1 \geq p_2$;*
- (ii) *better off if product H is ranked at the top than if L is, i.e., $U_H \geq U_L$;*
- (iii) *more likely to purchase a product at a given position if it is H than if it is L , i.e., $p_1 + p_2 \geq 1$;*
- (iv) *more likely to purchase product H than L if H is ranked above L , i.e., $p_1 > 1/2$.*

As search cost c increases, customers are:

- (v) *worse off, i.e., U_H and U_L are decreasing in c ;*
- (vi) *more likely to purchase a top-ranked product, i.e., p_1 is increasing and p_2 is decreasing in c .*

In the absence of search frictions, rankings exert no influence on which product customers purchase (they are always more likely to purchase product H) and how much utility they expect to derive, i.e., $p_1 = p_2 > 1/2$ and $U_H = U_L$ when $c = 0$ (this can be seen from Case 1.4 of Proposition 1 by letting $c = 0$). However, in the presence of search frictions (i.e., $c > 0$), the ranking effect can kick in. For a given product, customers are more likely to purchase it if it is placed at the top ($p_1 \geq p_2$ as in Corollary 1-(i)), and customers are better off when product H is ranked above product L ($U_H \geq U_L$ as in Corollary 1-(ii)). Moreover, for a given position (top or bottom), customers are more likely to purchase the product at the position if it is H than if it is L ($p_1 + p_2 \geq 1$ as in Corollary 1-(iii)). Corollary 1-(iv) immediately follows from combining Corollary 1-(i) and (iii): regardless of the search cost, if product H is ranked at the top, customers are more likely to purchase it than the one at the bottom, i.e., $p_1 > 1/2$.² The same cannot be said in general for a product H that takes the bottom position, i.e., p_2 can be smaller or larger than $1/2$, depending on the search cost, as we discussed earlier. As search becomes more costly, it is intuitive that customers are worse off (Corollary 1-(v)). Moreover, customers are more reluctant to search, and have an increasing chance of ending up with the product at the top. (Corollary 1-(vi)).

While our customer-search model is deliberately made very specific and stylized, we expect the properties distilled from the model as summarized in Corollary 1 to apply to much more general settings, given how intuitive they are. Our main insights will hold to the extent that these intuitive properties hold.

4. Random Ranking versus Sales-Based Ranking

The previous section sets up a model of customer search for a given snapshot of product rankings without specifying how the rankings are generated. In this section, we examine two conceptually intuitive ranking systems. The first one is the random ranking (§4.1); the second one is the sales-based ranking (§4.2), which serves as the basis for our model of brushing (to be formulated in

² Technically, combining $p_1 \geq p_2$ and $p_1 + p_2 \geq 1$ gives $p_1 \geq 1/2$. However, if $p_1 = 1/2$, then we must also have $p_2 = 1/2$, but based on Proposition 1, the case $p_1 = p_2 = 1/2$ cannot arise as a search outcome and thus is ruled out.

§5). We compare these two ranking systems in term of customer welfare, defined as the long-run average customer utility.

Neither of the ranking systems we consider requires any knowledge of customer preferences or product types on the part of the platform. Therefore, they are easy to implement and can be of practical appeal to the platform. Admittedly, products are rarely ranked purely in random order, but since the random-ranking system disregards sales volume, it naturally eliminates the incentive to brush, and therefore is a useful benchmark. That said, sales volume is often a predominant factor in determining the ordering of products: WSJ (2015b) cites a source that estimates sales volume accounts for 25% of how listings are generated on Alibaba, carrying more weight than any other known metrics. Furthermore, to the extent that other metrics are often correlated with sales volume, it may also serve as a first-order approximation for those more complex ranking algorithms.

The two ranking algorithms we consider contrast more sophisticated ranking algorithms that explicitly know customer preferences or product types. For example, an ideal personalized algorithm would know exactly what each customer wants and would only recommend the product that is the best for each individual. Alternatively, the platform could rank product H consistently above L for all customers. While these clairvoyant algorithms also do not take sales volume as an input (and thus would naturally eliminate brushing) and can be more beneficial to customers than the simple algorithms we consider, but they are much more difficult to implement because they involve tracking down every single customer or product, a formidable, if not impossible, task in practice given the massive number of customers and products on a given platform.

4.1. Benchmark: Random Ranking

Under the *random ranking*, in each period, the platform displays one of the two possible orderings of the products (product H above product L or product L above product H) with equal probability. We define customer welfare under the random ranking, W_R , to be

$$W_R \triangleq \frac{U_H + U_L}{2}, \quad (1)$$

where U_H and U_L are specified in Proposition 1. Note that since all periods are identical, customer welfare is equal to a customer's expected utility in each period before the ranking is realized. Alternatively, we can define the random ranking system as the following: the initial ranking is generated randomly at the beginning of the selling horizon, but once it is determined, it is fixed in all the subsequent periods. The two definitions yield equal customer welfare. The drawback of the random ranking is obvious: it treats the two products impartially even though customers would benefit from product H being placed at the top (i.e., $U_H \geq U_L$ by Corollary 1-(ii)).

4.2. Sales-Based Ranking

Under the *sales-based ranking*, the products are ranked in descending order of the number of units sold, i.e., the one with higher historical sales volume is placed above the one with lower sales volume. In each period, after a purchase, the sales volume of the purchased product is updated, and so are the rankings of the two products.

We formalize the evolution of the sales-based ranking as a generalized random walk on state space $\mathcal{S} \triangleq (\mathbb{Z} \setminus \{0\}) \cup \{\bar{0}, \underline{0}\}$, where any state $s \notin \{\bar{0}, \underline{0}\}$ denotes the difference in sales volume between products H and L ; state $\bar{0}$ (resp. $\underline{0}$) represents the case in which the two products have equal sales volume but product H (resp. product L) is at the top. To be clear, when $s > 0$ (resp. $s < 0$), the sales volume of product H (resp. product L) surpasses that of product L (resp. product H) by $|s|$ units and therefore is placed above it. We introduce Definition 1 that reloads some mathematical operators to accommodate states $\bar{0}$ and $\underline{0}$.

DEFINITION 1.

- (i) For state $s = \bar{0}$ or $s = \underline{0}$, $s + 1 = 1$ and $s - 1 = -1$; for state $s = 1$, $s - 1 = \bar{0}$; for state $s = -1$, $s + 1 = \underline{0}$.
- (ii) Let $f(s)$ be a function of $s \in \mathcal{S}$; $f(s)$ is increasing (resp. decreasing) in s if and only if $\dots \leq f(-2) \leq f(-1) \leq f(\underline{0}) \leq f(\bar{0}) \leq f(1) \leq f(2) \leq \dots$ (resp. $\dots \geq f(-2) \geq f(-1) \geq f(\underline{0}) \geq f(\bar{0}) \geq f(1) \geq f(2) \geq \dots$).
- (iii) $x^{\bar{0}} = x^{\underline{0}} \equiv 1, \forall x \in \mathbb{R} \setminus \{0\}$.

Let S_t be the state of the random walk at the beginning of period t , $t \in \mathbb{N}_0$. Thus, the state transition is given by

$$S_{t+1} = \begin{cases} S_t + 1, & \text{with probability } p_1 \\ S_t - 1, & \text{with probability } 1 - p_1 \end{cases}, \quad \forall S_t \in \{\bar{0}\} \cup \mathbb{Z}^+;$$

$$S_{t+1} = \begin{cases} S_t + 1, & \text{with probability } p_2 \\ S_t - 1, & \text{with probability } 1 - p_2 \end{cases}, \quad \forall S_t \in \{\underline{0}\} \cup \mathbb{Z}^-.$$

Figure 1 State Transition Diagram of the Random Walk Model of the Sales-Based Ranking

Figure 1 is the state transition diagram. Suppose the system is at state s at the beginning of a period. If the customer in that period ends up purchasing product H , then the sales volume of product H is incremented by one, and the system transitions into state $s + 1$ at the beginning of the next period; otherwise, it transitions into state $s - 1$. In particular, if the sales difference goes from $+1$ to 0 , then product H remains at the top, and the system state is $\bar{0}$; likewise, if the sales difference goes from -1 to 0 , then product L remains at the top, and the system state is $\underline{0}$. The transition probabilities are ranking-dependent. For any state $s \in \{\bar{0}\} \cup \mathbb{Z}^+$ (resp. $s \in \{\underline{0}\} \cup \mathbb{Z}^-$), product H (resp. product L) is at the top, and therefore the transition probability from s to $s + 1$ (the probability of product H being purchased) is p_1 (resp. p_2) as characterized in Proposition 1.

Next, we examine how the sales-based ranking behaves in the long run. Specifically, we are interested in two questions: (1) Will product H always rise to the top eventually? (2) How does it depend on the initial sales, if at all? To address these questions, we first introduce some notation. Given our random walk $\{S_t\}$, let $P(s) \triangleq \mathbb{P}\left\{\lim_{t \rightarrow \infty} S_t = +\infty \mid S_0 = s\right\}$ and $P'(s) \triangleq \mathbb{P}\left\{\lim_{t \rightarrow \infty} S_t = -\infty \mid S_0 = s\right\}$ be the probability that the random walk drifts to $+\infty$ and $-\infty$, respectively, if it starts at state $s \in \mathcal{S}$. We can interpret $P(s)$ as the probability that product H eventually settles on the top position if it initially surpasses product L in sales by s units. Lemma 1 characterizes $P(s)$ and $P'(s)$ subject to $p_1 + p_2 \geq 1$, $p_1 \geq p_2$, and $p_1 > 1/2$, based on the properties summarized in Corollary 1.³

LEMMA 1. *Under $p_1 + p_2 \geq 1$, $p_1 \geq p_2$, and $p_1 > 1/2$, random walk S_t either drifts to $+\infty$ or $-\infty$ as $t \rightarrow \infty$, i.e., $P(s) + P'(s) = 1$. In particular,*

(i) *when $p_2 \geq 1/2$, $P(s) \equiv 1, \forall s \in \mathcal{S}$;*

(ii) *when $p_2 < 1/2$,*

$$P(s) = \begin{cases} \begin{cases} 1 - \lambda_1 \left(\frac{1-p_1}{p_1}\right)^s, & \text{if } p_1 < 1, \\ 1, & \text{if } p_1 = 1, \end{cases} & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \begin{cases} \lambda_2 \left(\frac{1-p_2}{p_2}\right)^s, & \text{if } p_2 > 0, \\ 0, & \text{if } p_2 = 0, \end{cases} & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-, \end{cases}$$

where $\lambda_1, \lambda_2 \in (0, 1)$ are constants independent of s (with explicit expressions given in the appendix); moreover, $P(s)$ is increasing in s .

Lemma 1 shows that regardless of the initial sales, there will always be one product that eventually “wins out,” dominating the sales ranking with an unbeatable sales record, leading to a stable ranking (without oscillation) in the long run. Note that customers may still buy the product at the bottom even after the ranking stabilizes, but they are more likely to buy the winning product at the top, (stochastically) widening the gap between the two products’ cumulative sales over

³ We are able to characterize the asymptotic behavior of the random walk for any generic $p_1, p_2 \in [0, 1]$, but to be concise, we only present the cases relevant for our problem setting.

time. Furthermore, Lemma 1-(i) states that if customers are more likely to purchase product H regardless of its ranking (i.e., $p_1 > 1/2$ and $p_2 \geq 1/2$), then product H will inevitably claim the top spot in the long run, even when it lags behind in sales volume initially. In other words, customers will always collectively “discover” product H . Understandably, this case arises—with reference to our earlier results in Proposition 1—when the search cost is relatively low (Cases 1.3 and 1.4 of Proposition 1).

However, when the search cost is relatively high (Cases 1.1 and 1.2 of Proposition 1), a bottom-ranked product H has a lower chance of being chosen than the one ranked above it (i.e., $p_2 < 1/2$), and Lemma 1-(ii) shows that in this case, there is no guarantee for product H to eventually land on the top position. In fact, the opposite might occur: product L might wind up with more sales and a higher ranking. Customers’ possible inability to collectively discover product H may sound surprising, but it is a natural outcome of search frictions. Consider Case 1.1 of Proposition 1, in which customers are inflicted by a high search cost and thus always purchase the product at the top without searching the second one. In this scenario, the rankings are fixed throughout the selling horizon.

More generally, Lemma 1-(ii) indicates that when $p_2 < 1/2$, how the two products are ranked relative to each other in the long run is sensitive to the initial sales difference between the two. The more initial sales advantage the product H has over product L , the more likely product H will eventually dominate the rankings. As an example, consider Case 1.2 of Proposition 1, in which customers are subject to an intermediately high search cost. In this case, if product H is initially ranked at the top, then it remains at the top in every period; nevertheless, if product H is initially buried underneath product L because it falls short of sales, then the more it falls short, the less likely, albeit still possible, the rankings will ultimately flip.

4.2.1. Customer Welfare

Recall from Corollary 1-(ii) that customers are better off if product H is placed at the top than otherwise, i.e. $U_H \geq U_L$. However, Lemma 1 shows that the sales-based ranking system may fail to push product H to the top. As a consequence, it is unclear whether customers are better off under the sales-based ranking than under the random ranking. To investigate this question, we first define customer welfare, W_S , under the sales-based ranking:

$$W_S \triangleq \frac{1}{2} \left(P(\bar{0})U_H + [1 - P(\bar{0})]U_L + P(\underline{0})U_H + [1 - P(\underline{0})]U_L \right), \quad (2)$$

where U_H and U_L are specified in Proposition 1; $P(\bar{0})$ and $P(\underline{0})$, characterized in Lemma 1. At the beginning of the selling horizon, both products have zero sales, and the initial ranking is randomly determined, i.e., the initial state S_0 can be either $\bar{0}$ or $\underline{0}$ with equal probability. If the initial state

is $\bar{0}$, i.e., product H is at the top, then with probability $P(\bar{0})$, it remains at the top in the long run, in which case, the expected utility for an arriving customer is U_H ; with probability $1 - P(\bar{0})$, product H falls to the bottom in the long run, in which case, the expected utility for an arriving customer is U_L . A similar argument can be made for an initial state of $\underline{0}$, and hence the customer-welfare expression W_S in Equation (2) for the sales-based ranking. Next, we compare customer welfare under the sales-based ranking, W_S , with that under the random ranking, W_R (as defined in Equation (1)).

THEOREM 1. *The sales-based ranking weakly improves customer welfare relative to the random ranking, i.e., $W_S \geq W_R$. In particular, customer welfare is strictly improved ($W_S > W_R$) if and only if $c < \gamma(v_H - v_0)$.*

Although the sales-based ranking may end up placing product L at the top, Theorem 1 shows that it still weakly improves customer welfare, relative to the random ranking. Specifically, when the search cost is high enough (i.e., $c \geq \gamma(v_H - v_0)$ as in Case 1.1 of Proposition 1), customers cannot afford to search the second product, and always purchase the one at the top; hence the sales-based ranking system will effectively be identical to a random ranking system. When the search cost is not too high (i.e., $c < \gamma(v_H - v_0)$ as in Cases 1.2 through 1.4 of Proposition 1), the sales-based ranking system makes customers strictly better off. Since customers never fail to push product H to the top when the search cost is relatively low (Cases 1.3 and 1.4 of Proposition 1), it is immediate that the sales-based ranking is beneficial to customers. When the search cost is intermediately high (Case 1.2 of Proposition 1), if product H is at the top initially, then it remains at the top; if not, then there is still a chance for it to move to the top; as a result, customers once again prefer the sales-based ranking to the random ranking.

Our result implies that product H receives a favorable bias in the sales-based ranking system. The simplicity of such a system combined with its ability to improve customer welfare offers a potential explanation for why it is so commonly used. However, this welfare gain is established with the caveat that sellers do not react to the ranking systems. In this next section, we will explore how this result is affected when sellers are strategic.

5. Brushing

Under the random ranking, the sellers have no incentive to brush because inflating sales does not affect their rankings and thus does not change customers' purchase probabilities. By contrast, the sales-based ranking creates an incentive for brushing because a higher sales figure implies a higher ranking and a higher purchase probability, which ultimately translates into higher profit.

5.1. A Brushing Game

We now formulate a brushing game between the two sellers in the sales-based ranking system. Let the reward each seller earns from selling each unit of his product be R ; the cost he incurs by brushing one unit, c_B . The brushing cost captures two aspects: (1) the physical cost of processing fake transactions and compensating third-party brushing professionals. (2) the difficulty of brushing in light of the regulatory environment: heftier penalties and tougher audits make brushing harder (namely, increasing the brushing cost) for sellers. We deliberately assume symmetry in unit reward and brushing cost across the two sellers such that the only difference between the two is the potential difference in the net valuation their products provide. This will help us tease out the minimal nontrivial conditions for the two sellers to differ in their brushing strategies.

In our base model, each seller chooses between two brushing quantities, 0 (zero) or $Q \in \mathbb{N}$ orders, in the initial period. Specifically, seller $i \in \{H, L\}$ determines strategy $\beta_i \in [0, 1]$, the probability he assigns to brushing Q orders, and correspondingly, $1 - \beta_i$, the probability with which seller i chooses not to brush. If $\beta_i \in \{0, 1\}$, then it is a pure brushing strategy; any $\beta_i \in (0, 1)$ is a mixed strategy. Strategy β_i captures the intensity of brushing: for example, $\beta_L > \beta_H$ implies seller L brushes (stochastically) more than seller H .

We have made two simplifications for tractability purposes. First, we consider a one-shot brushing game between the two forward-looking sellers who front-load all the fake orders to time zero. The one-shot model is a tractable first cut of the problem, yet it still reveals some fundamental characteristics of the underlying strategic interactions between the sellers. We believe it is a reasonable starting point of analysis. Second, in our base model, we restrict possible quantities of brushing to $\{0, Q\}$, with Q being a model primitive (we do allow for mixed strategies over $\{0, Q\}$). Doing so gives us the simplest setting to cleanly identify the first-order effect of brushing. We will relax this restriction by considering more general brushing strategies in §6.2.

Each seller maximizes his total expected discounted profit under a common discount factor $\delta \in (0, 1)$. For $i \in \{H, L\}$, let $\pi_i(\beta_H, \beta_L)$ denote seller i 's total expected discounted profit, which is equal to the total expected discounted reward from authentic sales less the total brushing cost. Let $V_i(s)$ be the total expected discounted reward of seller $i \in \{H, L\}$ when the initial state after brushing is $s \in \mathcal{S}$. Thus, for seller $i \in \{H, L\}$,

$$\pi_i(\beta_H, \beta_L) \triangleq \beta_H(1 - \beta_L)V_i(Q) + \beta_L(1 - \beta_H)V_i(-Q) + [\beta_H\beta_L + (1 - \beta_H)(1 - \beta_L)] \cdot \frac{V_i(\bar{0}) + V_i(\underline{0})}{2} - \beta_i c_B Q.$$

Based on the possible strategies of the sellers, three initial states can arise after brushing: (1) seller H brushes Q units and seller L , 0 units, which occurs with probability $\beta_H(1 - \beta_L)$ and leads to an initial state $S_0 = Q$; (2) seller L brushes Q units and seller H , 0 units, which occurs with

probability $\beta_L(1 - \beta_H)$ and leads to an initial state $S_0 = -Q$; (3) both sellers brush the same units (i.e., either 0 or Q), which occurs with probability $\beta_H\beta_L + (1 - \beta_H)(1 - \beta_L)$ and leads to an initial state $s \in \{\bar{0}, \underline{0}\}$ with equal probability.

The total expected discounted reward of seller i , $V_i(s)$, satisfies the following recursive equations (which are set up by conditioning on the first step):

$$\begin{cases} V_H(s) = p(s)[R + \delta V_H(s+1)] + [1 - p(s)]\delta V_H(s-1), \\ V_L(s) = p(s)\delta V_L(s+1) + [1 - p(s)][R + \delta V_L(s-1)], \end{cases} \quad \text{where } p(s) = \begin{cases} p_1, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ p_2, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases}$$

Recall that p_1 and p_2 are the purchase probabilities characterized in Proposition 1. It is clear from the recursive equations that each time the system moves from state s to $s+1$ (resp. $s-1$), seller H (resp. seller L) earns reward R . A brushing strategy profile (β_H, β_L) is a Nash equilibrium if

$$\beta_H \in \arg \max_{\beta'_H} \pi_H(\beta'_H, \beta_L), \quad \beta_L \in \arg \max_{\beta'_L} \pi_L(\beta_H, \beta'_L).$$

5.2. Brushing Incentives and Equilibrium

The incentive for brushing is twofold: (1) to stay ahead of competition, termed the *offensive incentive*; (2) to avoid lagging behind, termed the *defensive incentive*. To disentangle these two aspects of strategic thinking, we introduce the marginal gain in reward from brushing (*excluding* the change in brushing costs) for seller $i \in \{H, L\}$ when neither seller brushes, denoted by M_i , which sheds light on the offensive incentive; and the marginal loss of reward from not brushing (*excluding* the change in brushing costs) for seller i when both sellers brushes, denoted by M'_i , which sheds light on the defensive incentive. For seller $i \in \{H, L\}$, let β_{-i} denote the other seller's strategy. Thus, M_i and M'_i are given by:

$$M_i \triangleq \pi_i(\beta_i = 1, \beta_{-i} = 0) - \pi_i(\beta_i = 0, \beta_{-i} = 0) + c_B Q = \begin{cases} V_H(Q) - [V_H(\bar{0}) + V_H(\underline{0})]/2, & \text{if } i = H, \\ V_L(-Q) - [V_L(\bar{0}) + V_L(\underline{0})]/2, & \text{if } i = L. \end{cases}$$

$$M'_i \triangleq \pi_i(\beta_i = 1, \beta_{-i} = 1) - \pi_i(\beta_i = 0, \beta_{-i} = 1) - c_B Q = \begin{cases} [V_H(\bar{0}) + V_H(\underline{0})]/2 - V_H(-Q), & \text{if } i = H, \\ [V_L(\bar{0}) + V_L(\underline{0})]/2 - V_L(Q), & \text{if } i = L. \end{cases}$$

Lemma 2 below gives structural properties of the total expected discounted reward $V_i(s)$, and those of M_i and M'_i for both sellers. (We derive closed-form expressions of $V_i(s)$ in the appendix.)

LEMMA 2. Under $p_1 + p_2 \geq 1$, $p_1 \geq p_2$, and $p_1 > 1/2$,

(i) $V_H(s)$ is increasing in s and $V_L(s)$ is decreasing in s ;

(ii) $M_L \geq M_H$ and $M'_L \leq M'_H$ with both inequalities strict when $p_1 + p_2 > 1$ and $p_1 > p_2$.

Lemma 2-(i) confirms the intuition that the bigger the initial sales advantage a seller has over his competitor, the higher the seller's total expected discounted reward will be. Thus, both sellers

have an incentive to brush in order to get a head start. Nonetheless, Lemma 2-(ii) shows subtle asymmetry in brushing incentives across the two sellers. Specifically, when neither seller brushes, seller L has a stronger incentive to unilaterally start brushing because he benefits more from doing so (i.e., $M_L \geq M_H$). However, when both sellers brush, seller L has a stronger incentive to unilaterally stop brushing because he has less to lose (i.e., $M'_L \leq M'_H$). Moreover, Lemma 2-(ii) reveals that such asymmetry (in the strict sense) only occurs when the search cost is intermediate (Cases 1.2 and 1.3 of Proposition 1, which satisfy $p_1 + p_2 > 1$ and $p_1 > p_2$).

Here is the intuition. When the search cost is intermediate, customers conduct some search but are still susceptible to the product rankings, whose evolution favorably biases seller H . In the absence of brushing, seller L is more tempted to brush to counteract his disadvantage in the sales-based ranking. On the other hand, when both brush, their fake orders are offset, and seller L finds brushing less fruitful, again because his inherent disadvantage implies there is not as much to lose even if he stops brushing. Hence, brushing by seller L (resp. H) is more driven by the offensive (resp. defensive) incentive. Building on Lemma 2, Proposition 2 characterizes the equilibrium strategy of the two sellers.

PROPOSITION 2. *The sellers' equilibrium brushing strategies are as follows: there exist unique thresholds $\overline{C}$ and $\underline{C}$ on brushing cost c_B with $\overline{C} \geq \underline{C} \geq 0$ such that*

(i) *If $0 < c_B < \underline{C}$, then $(\beta_H, \beta_L) = (1, 1)$ is the unique equilibrium, which is a dominant-strategy equilibrium;*

(ii) *If $c_B > \overline{C}$, then $(\beta_H, \beta_L) = (0, 0)$ is the unique equilibrium, which is a dominant-strategy equilibrium;*

(iii) *If $\underline{C} < c_B < \overline{C}$, then (β_H, β_L) is the unique (mixed-strategy) equilibrium with $\beta_H, \beta_L \in (0, 1)$; additionally, β_H (resp. β_L) is decreasing (resp. increasing) in c_B ;*

(iv) *If $c_B = \underline{C}$, the set of equilibria is (β_H, β_L) with $\beta_H = 1$ and $\beta_L \in [0, 1]$; all the equilibria generate the same expected profit for seller L ;*

(v) *If $c_B = \overline{C}$, the set of equilibria is (β_H, β_L) with $\beta_H = 0$ and $\beta_L \in [0, 1]$; all the equilibria generate the same expected profit for seller L ;*

Moreover, $\overline{C}$ and $\underline{C}$ are both increasing in search cost c and (1) $\overline{C} = \underline{C} = 0$ if $c \leq \gamma(v_L - v_0)$; (2) $\overline{C} = \underline{C} > 0$ if $c \geq \gamma(v_H - v_0)$; (3) $\overline{C} > \underline{C} > 0$ if $\gamma(v_L - v_0) < c < \gamma(v_H - v_0)$.

Figure 2 illustrates Proposition 2 by showing the equilibrium brushing probabilities of both sellers as a function of brushing cost c_B with the solid (resp. dashed) line representing seller H 's strategy β_H (resp. seller L 's strategy β_L).

At a high level, the equilibrium structure identified by Proposition 2 is quite intuitive. Parts (i) through (iii) of Proposition 2 show that when the brushing cost is sufficiently high (i.e., $c_B > \overline{C}$),

Figure 2 Equilibrium Brushing Probabilities of the Two Sellers

Note. $\delta = 0.9$, $R = 1$, $v_H = 7$, $v_L = 4.5$, $v_0 = 3$, $Q = 20$, $\gamma = 0.8$, $c = 2.6$.

there is *no brushing*. When the brushing cost is sufficiently low (i.e., $c_B < \underline{C}$), both sellers engage in *full brushing*, namely, both faking Q orders. When the brushing cost is intermediate (i.e., $\bar{C} < c_B < \underline{C}$), both sellers engage in *partial brushing* by randomizing over faking zero and Q orders. However, at a more granular level, the equilibrium structure is subtle in terms of how the two sellers' differ in their brushing intensity. In particular, under an intermediate brushing cost (i.e., $\bar{C} < c_B < \underline{C}$), seller H brushes less as brushing becomes more costly (which is intuitive), but seller L brushes more (which is counter-intuitive). Moreover, seller L brushes more than seller H when the brushing cost is intermediately high, but less when the brushing cost is intermediately low.

To understand why, it is best to investigate what happens when the brushing cost is at the two thresholds, namely, $c_B \in \{\underline{C}, \bar{C}\}$. According to parts (iv) and (v) of Proposition 2, these are the only two cases in which multiple equilibria exist. As stated earlier, neither seller brushes under a high brushing cost. When the brushing cost falls and reaches the upper threshold $\bar{C}$, seller H still strictly prefers no-brushing, yet seller L becomes indifferent between brushing and not brushing and thus may choose any possible brushing intensity. This type of equilibria can indeed be sustained because seller L has a stronger offensive incentive, as shown in Lemma 2. On the other end of the spectrum, both sellers brush under a low brushing cost. When the brushing cost rises and reaches $\underline{C}$, seller H still strictly prefers full-brushing, yet seller L becomes indifferent between brushing and not brushing and thus may choose any possible brushing intensity. This type of equilibria can indeed be sustained because seller L 's defensive incentive is not as strong, as shown in Lemma 2. Given seller L 's disadvantage in the sales-based ranking system, he has to determine his brushing strategy by circumventing that of his opponent. Consequently, when the brushing cost is intermediate, (i.e., $\bar{C} < c_B < \underline{C}$), an increase in brushing cost decreases seller H 's brushing intensity, and seller L , in turn, captures the opportunity and brushes more.

In addition to the sellers' brushing cost, Proposition 2 shows that customers' search cost also plays a critical role in shaping the brushing equilibrium. When the search cost is sufficiently low (Case 1.4 of Proposition 1), neither seller brushes regardless of how inexpensive brushing is. This is because at such low search frictions, customers are immune to the ranking effect (in the sense that $p_1 = p_2$) and can always find out their favorite product. Hence, manipulating the ranking through brushing would be futile for the sellers. Following this logic, a higher search cost induces more brushing (i.e., both thresholds, $\underline{C}$ and $\overline{C}$, increase in the search cost). When the search cost is sufficiently high (Case 1.1 of Proposition 1), the two sellers always adopt the same brushing strategy (either full brushing or no brushing, but not partial brushing in which they can have different brushing intensities). In this case, as explained earlier, customers do not effectively search and always pick whichever is ranked at the top, making the two sellers symmetric (because they have equal chance of being placed at the top initially), and therefore, their equilibrium brushing strategies are always identical.

Note that the parameter region that generates the partial brushing equilibrium is arguably the most interesting and relevant region for practical purposes. Brushing is widespread, so we are most likely not situated in an environment that sustains the no-brushing equilibrium. On the other hand, the rankings will be unaffected if all sellers brush the same amount, which would be the case in the full-brushing equilibrium, but this also sounds somewhat implausible.

5.3. Customer Welfare

In this subsection, we study the welfare implications of brushing. Let W_B denote customer welfare in the sales-based ranking subject to brushing:

$$W_B \triangleq [\beta_H \beta_L + (1 - \beta_H)(1 - \beta_L)] W_S + \beta_H(1 - \beta_L)[P(Q)U_H + (1 - P(Q))U_L] + \beta_L(1 - \beta_H)[P(-Q)U_H + (1 - P(-Q))U_L], \quad (3)$$

where β_H and β_L are characterized in Proposition 2; W_S , defined in Equation (2); U_H and U_L , specified in Proposition 1; $P(Q)$ and $P(-Q)$, pinned down in Lemma 1. Note that when both sellers engage in full brushing (i.e., $\beta_H = \beta_L = 1$) or neither brushes (i.e., $\beta_H = \beta_L = 0$), customer welfare reduces to that under the sales-based ranking system without brushing ($W_B = W_S$). When seller H brushes Q orders yet seller L does not brush, we would have $W_B = P(Q)U_H + (1 - P(Q))U_L$; it immediately follows from the monotonicity of $P(\cdot)$ (Lemma 1) and $U_H \geq U_L$ (Corollary 1-(ii)) that the presence of brushing weakly improves customer welfare ($W_B \geq W_S$). At the other extreme, when seller L brushes Q orders yet seller H does not brush, customer welfare would weakly deteriorate ($W_B \leq W_S$).

Combining Propositions 1-2 and Theorem 1, we compare customer welfare in three systems: customer welfare under the benchmark random-ranking system, W_R , as defined in Equation (1), customer welfare under the sales-based ranking system assuming sellers do not brush, W_S , as defined in Equation (2), and customer welfare under the sales-based ranking system subject to brushing, W_B , as defined in Equation (3). We also examine the impact of brushing cost c_B on customer welfare W_S . Theorem 2 summarizes our results.

THEOREM 2. *Customer welfare under brushing, W_B , has the following properties:*

- (i) *when $c \in (0, \gamma(v_H - v_L)] \cup [\gamma(v_H - v_0), +\infty)$, then $W_B = W_S$;*
- (ii) *when $c \in (\gamma(v_H - v_L), \gamma(v_H - v_0))$, then $W_B = W_S$ for $c_B \in (0, \underline{C}) \cup (\overline{C}, +\infty)$, and W_B is decreasing in c_B for $c_B \in (\underline{C}, \overline{C})$; in addition, there exist $\overline{C}'$ and $\underline{C}'$ with $\overline{C} \geq \overline{C}' > \underline{C}' > \underline{C}$ such that*
 - (a) *$W_B > W_S > W_R$ if and only if $\underline{C} < c_B < \underline{C}'$;*
 - (b) *$W_S > W_B > W_R$ if and only if $\underline{C}' < c_B < \overline{C}'$;*
 - (c) *$W_S > W_R > W_B$ if and only if $\overline{C}' < c_B < \overline{C}$.*

In particular, there exists $\bar{\gamma} \in (0, 1)$ such that $\overline{C}' < \overline{C}$ if and only if $\gamma > \bar{\gamma}$.

Figure 3 Comparison of Customer Welfare Under Three Systems, Random Ranking (W_R), Sales-Based Ranking Without Brushing (W_S) and with Brushing (W_B)

Note. $\delta = 0.9$, $R = 1$, $v_H = 7$, $v_L = 4.5$, $v_0 = 3$, $Q = 20$, $\gamma = 0.8$.

The gist of Theorem 2 is illustrated by Figure 3. Theorem 2-(i) shows that when customers' search cost is either relatively low or sufficiently high, brushing in the sales-based ranking system

does not impact customer welfare (i.e., $W_B = W_S$). When the search cost is sufficiently low (Case 1.4 of Proposition 1), as we explained previously (Proposition 2), neither seller brushes, and therefore $W_B = W_S$. Likewise, when the search cost is sufficiently high (Case 1.1 of Proposition 1), we know from Proposition 2 that there is either no brushing or full brushing, in which case, countervailing brushing from the two sellers offsets each other; once again, we have $W_B = W_S$. When the search cost is intermediately low (Case 1.3 of Proposition 1), the equilibrium brushing intensity of the two sellers can differ, but we still have $W_B = W_S$. In this case, $p_2 > 1/2$, which implies that seller H will eventually claim the top position in the long run, regardless of how brushing perturbs the initial state (see Lemma 1). Therefore, customer welfare remains unaffected.

The most interesting scenario arises when customers' search cost is intermediately high (Case 1.2 of Proposition 1) and the sellers' brushing cost is intermediate ($\underline{C} \leq c_B \leq \bar{C}$), as illustrated by the light-gray area in the third region from the left in Figure 3. Theorem 2-(ii) shows that in this case, a lower brushing cost actually leads to higher customer welfare W_B (as illustrated by Figure 4). This follows from Proposition 2: with a lower brushing cost, seller H brushes more while seller L brushes less, and therefore, customers are better off.

Building on this result, Theorem 2-(ii) further suggests that when the brushing cost is high enough in this scenario (i.e., $c_B \in (\bar{C}', \bar{C})$), (because seller L brushes substantially more than seller H) brushing makes customers even worse off than what they would be in the random ranking system (i.e., $W_B < W_R$)—let alone the sales-based ranking system without brushing—as long as the fit probability is high enough.⁴ On the other hand, when the brushing cost is low enough in

⁴ When product L is at the top, a high fit probability predisposes customers to product L by dampening their incentive to search, making brushing particularly harmful to customer welfare.

this scenario (i.e., $c_B \in (\underline{C}, \underline{C}')$), customers can benefit from brushing (i.e., $W_B > W_S$) as seller H far outstrips seller L in brushing intensities.

Theorem 2 reveals the unintended consequences of the sale-based ranking. While it is meant to assist customers with product discovery and improve customer welfare (Theorem 1), it may also trigger sellers' strategic brushing behavior, which could completely undo and even invert its welfare-improvement effect. However, the sales-based ranking does not always backfire: under certain circumstances, the presence of brushing can surprisingly further improve customer welfare. Moreover, while conventional wisdom contends that making brushing harder would deter such behavior and benefit customers, we find that it may only deter seller H while prompting seller L to brush more, thereby reducing customer welfare. Theorem 2 has managerial implications for platforms concerned about customer welfare. First, it might not always be a good idea to eliminate brushing altogether. Second, given that brushing is difficult to eradicate, escalating the combat against brushing may only hurt customers, and therefore, a laissez-faire policy might be more advisable instead.

We have studied the impact of the sellers' brushing cost on customer welfare. Next, Theorem 3 examines the impact of customers' search cost.

THEOREM 3. *As search cost c decreases,*

(i) Customer welfare under the random ranking, W_R , and that under the sales-based ranking, W_S , both increase;

(ii) Customer welfare under brushing, W_B , can decrease;
in particular, W_B under c is less than W'_B under c' for any $c < c'$ if $c \in (\gamma(v_H - v_L), \gamma(v_H - v_0))$, $c' > \gamma(v_H - v_0)$, $c_B \in (\overline{C}', \overline{C})$ and fit probability γ is sufficiently high.

By conventional wisdom, a lower search cost facilitates customers' discovery of more desirable products and thus benefits customers. Recall from Corollary 1-(v) that it indeed holds for a snapshot of product rankings. Theorem 3-(i) further extends it to the cases of the random ranking system and the sales-based ranking system without brushing (see the solid and dotted lines in Figure 5). However, Theorem 3-(ii) shows that the intuition may not always be valid in the sales-based ranking system with brushing; specifically, a lower search may exacerbate customer welfare (see the dashed line in Figure 5).

In its standalone form, this result may sound striking: sellers brush to manipulate the rankings, which make a difference precisely owing to customers' search frictions; hence, one would expect lowering search costs can discourage sellers from brushing and improve customer welfare. Indeed, when the search cost is low enough, as we argued in Proposition 2, neither seller brushes. However, what this line of reasoning is missing is that when it comes to customer welfare, not only does the

Figure 5 Customer Welfare vs. the Search Cost

Note. $\delta = 0.9$, $R = 1$, $v_H = 7$, $v_L = 4.5$, $v_0 = 3$, $Q = 20$. $\gamma = 0.8$, $c_B = 0.22$.

absolute magnitude of brushing matters; the *relative* intensity and the underlying sales-updating dynamics also do. It is possible that under a high search cost (Case 1.1 of Proposition 1 to be specific), both sellers engage in full brushing and therefore their effects on customer welfare cancel out; under a lower search cost (Case 1.2 of Proposition 1), both sellers do brush less (they move to a partial-brushing equilibrium), but seller H brushes even less than seller L does, leading to lower customer welfare. Theorem 3 has managerial implications for platforms: making search easier for customers is not necessarily a panacea that always improves customer welfare; one must take into account sellers' strategic response as well.

6. Extensions and Further Thoughts

In this section, we consider two extensions, heterogeneous search costs and more general brushing strategies; we also discuss the connection between brushing and the well-studied position auction.

6.1. Heterogeneous Search Costs

In the base model, all customers have the same search cost. In this subsection, we consider an extension in which customers are heterogeneous in search costs. Specifically, we assume an $\alpha \in (0, 1)$ fraction of customers are informed and face zero search costs; the other $1 - \alpha$ fraction of customers are still uniformed and incur search cost c for searching the second product. Unlike the uniformed customers, the informed ones do not necessarily search from the top and always choose the product that give them the higher net valuation. In this modified model, all the structural properties identified in Corollary 1 remain valid. Since our random-walk model is developed using general purchase probabilities p_1, p_2 (see Lemmas 1 and 2) satisfying properties in Corollary 1, we can readily accommodate the case of heterogeneous search costs by modifying the expressions of

p_1, p_2 in Proposition 1. We show our main insights hold as long as α is not too large (a sufficiently large α would trivially nullify the relevance of product rankings). Interestingly, in this case, not only can our most interesting results (in terms of the impact of brushing on customer welfare) arise when the search cost is intermediately high (Case 1.2 of Proposition 1), as in the base model, but they may also arise when the search cost is high (Case 1.1 of Proposition 1). Thus, considering heterogeneous search costs further strengthens the relevance of our results. The interested reader is referred to Appendix C for details.

6.2. More General Brushing Strategy

In the base model, the sellers choose the probabilities they assign to brushing Q orders or not brushing at all. In this subsection, we take one step further by considering more general brushing strategies whereby sellers can choose among $\{0, Q, 2Q\}$ units to brush (with mixed strategies allowed). This extended model captures low, moderate and high brushing scenarios. While the equilibrium conditions are more complex, we numerically find our main insights robust to such an extension. Figure 6 shows one such example; it is a replication of Figure 4, but now, both sellers are allowed to brush among $\{0, Q, 2Q\}$, as opposed to $\{0, Q\}$ in the original figure.

We make the following observations from Figure 6. First, when the brushing cost is high enough, neither seller brushes; when the brushing cost is low enough, both sellers brush $2Q$ units (full brushing); in either case, customer welfare is not altered by the presence of brushing. Second, when the brushing cost is intermediate, the sellers engage in partial brushing; customer welfare under brushing can be lower than that in the random ranking system, but under some other

circumstances, can also be higher than that in the sales-based ranking system without brushing. Third, customer welfare is non-monotone in the brushing cost, and in particular, a higher brushing cost can dampen customer welfare. Note that these observations are qualitatively consistent with our analytical results derived from the base model.

On a more granular level, since the brushing strategies are more general, the equilibrium structure also becomes richer. Specifically, three types of mixed-strategy equilibria (in addition to two types of pure-strategy equilibria) emerge in this case. When the brushing cost is sufficiently high, neither brushes. As the brushing cost decreases, both sellers start randomizing between brushing 0 and Q units; then they mix among all the three possible brushing quantities, 0, Q , $2Q$; and finally, seller H mixes between Q and $2Q$ units, whereas seller L mixes between 0 and $2Q$ units, before both brush $2Q$ units. Clearly, even in this three-point-brushing model, the equilibrium strategy is highly intricate and not quite amenable to tractable analytical characterization.

Part of the complications is the lack of pure-strategy equilibria in non-extreme cases. We examine whether this observation is a general property of the brushing game. To do that, we consider an even more general case in which the brushing quantities can be any natural numbers. Let $Q_H, Q_L \in \mathbb{N}_0$ be the quantities that seller H and seller L brush, respectively. Thus, with slight abuse of notation, the total expected discounted profit for seller $i \in \{H, L\}$ is:

$$\pi_i(Q_H, Q_L) \triangleq \begin{cases} [V_i(\bar{0}) + V_i(\underline{0})]/2 - c_B Q_i, & \text{if } Q_H = Q_L, \\ V_i(Q_H - Q_L) - c_B Q_i, & \text{otherwise.} \end{cases}$$

A pure-strategy Nash equilibrium (Q_H, Q_L) jointly satisfies the following conditions:

$$Q_H \in \arg \max_{Q_H} \pi_H(Q_H, Q_L), \quad Q_L \in \arg \max_{Q_L} \pi_L(Q_H, Q_L).$$

PROPOSITION 3. *Under $p_1 + p_2 \geq 1$, $p_1 \geq p_2$, and $p_1 > 1/2$, when both sellers can brush any quantity of their choice, there exists $C_0 \geq 0$ such that neither seller brushes (i.e., $Q_H = Q_L = 0$) if and only if $c_B \geq C_0$; additionally,*

(i) *if $p_1 + p_2 > 1$, then $(Q_H, Q_L) = (0, 1)$ is an equilibrium when $c_B = C_0$ and there are no pure strategy equilibria for $c_B < C_0$;*

(ii) *if $p_1 + p_2 = 1$, then $(Q_H, Q_L) \in \{(0, 1), (1, 0), (1, 1)\}$ are equilibria when $c_B = C_0$ and there are no pure strategy equilibria for $c_B < C_0$;*

(iii) *$C_0 > 0$ if and only if $p_1 > p_2$.*

Proposition 3 suggests that there exist no pure strategy brushing equilibria except under knife-edge cases (i.e., $c_B = C_0$). This is because if a seller were to brush, he would only brush one more unit than his competitor, but his competitor follows exactly the same strategic thinking, thus leading to the non-existence of pure-strategy equilibria. This calls for a cumbersome analysis of

mixed strategy equilibria in which each seller's strategy in general is a probability distribution over the support of all natural numbers. By contrast, our base model of two-point-brushing (along with its three-point extension) is a tractable modeling alternative.

6.3. Connection with the Position Auction

Fundamentally, brushing is a ranking competition among sellers, and therefore it has a natural connection with the position auction in sponsored advertising (e.g., Jerath et al. 2011). We comment on the differences of their underlying mechanisms (beyond the obvious difference in application contexts). To be concrete, we adapt our model to consider a second-price position auction, in which the seller who bids higher (the winner) gets the top position and the other seller (the loser) gets the bottom position. The winner pays the bid of the losing seller, whereas the latter does not pay. The one-shot auction is conducted at the beginning of the selling horizon. Based on the auction outcome, the winning seller pays upfront in lump sum; the rankings of the two sellers is determined and remains fixed throughout the selling horizon.

For seller H (resp. L), if he is placed at the top, his total expected discounted reward will be $p_1 R / (1 - \delta)$ (resp. $(1 - p_2) R / (1 - \delta)$); if he is placed at the bottom, his total expected discounted reward will be $p_2 R / (1 - \delta)$ (resp. $(1 - p_1) R / (1 - \delta)$). Therefore, for each seller, their willingness to pay (valuation) for the top position is the difference between the reward to be gained from the top position and that from the bottom position, which, for both sellers, is

$$\frac{(p_1 - p_2) R}{1 - \delta}.$$

It is well known that in a second-price auction, a weakly dominant strategy is to truthfully bid one's willingness to pay⁵. Since the two sellers have equal valuations, they bid the same amount (equal to $(p_1 - p_2) R / (1 - \delta)$) in equilibrium.

The bidding equilibrium in the position auction contrasts the brushing equilibrium in the brushing game. In the position auction, the two sellers submit the same bid when they earn the same reward per sale, whereas in the brushing game, the two sellers may still brush differently in equilibrium. This is because in the position auction, each seller attaches the same relative value to the top position, and their willingness to pay is independent of the other seller's bidding strategy, whereas in the brushing game, the two sellers value the top position (relative to the bottom one) differently, and their valuation depends on how the other seller brushes (see Lemma 2-(ii)). The key distinction is that the rankings in the position auction are purely based on the relative magnitude of the bids and once determined, are fixed over the selling horizon, whereas the rankings in the brushing game

⁵ The position auction here essentially sells a single item (the top position), and hence the truthful bidding property holds.

evolve over time. Hence, the strategic thinking behind brushing is more sophisticated, and due to the differential treatment the two sellers may receive in the sales-based ranking, they may also have disparate incentives in manipulating the rankings.

7. Conclusion and Discussion

This paper studies an emerging phenomenon called “brushing” on e-commerce platforms, whereby sellers place fake orders of their own products to inflate sales and boost rankings. Two factors engender brushing. First, customers face search frictions and tend to be fixated only on the few most prominent products at the top of search results. Second, online marketplaces often generate rankings based on historical sales: a product that sells well in the past obtains a higher placement, which, in turn, drives more future sales. We show that such a well-intentioned sales-based ranking system always improves customer welfare in principle (i.e., when sellers are not strategic) relative to a case in which products are randomly ranked, but it is also susceptible to brushing as a means of manipulation. Our paper sheds light on the diverging brushing incentives of high-quality sellers and low-quality ones. The high-quality seller resorts to brushing more as a defense mechanism, whereas the low-quality one uses brushing to take the offensive. We find that in the presence of brushing, the sales-based ranking can backfire and achieve even lower customer welfare than the random ranking. Under the right circumstances, though, brushing can surprisingly further raise customer welfare as the high-quality seller brushes substantially more than his low-quality opponent.

Our paper generates managerial implications for platforms. A stiffer crackdown on brushing may surprisingly make customers worse off. Thus, a *laissez-faire* approach to brushing may be more beneficial. Moreover, while brushing is partially attributed to the presence of search frictions, making search easier for customers may, in fact, exacerbate brushing’s deleterious effect on customer welfare. Thus, platforms should not treat lowering search frictions as an infallible remedy, but should also scrutinize the consequences of sellers’ strategic response.

Below, we discuss several modeling assumptions and future research directions. In our model, we implicitly assume that customers do not observe the products’ sales volume. The lack of such information implies that customers can neither use it as a signal for them to infer product type nor discount it because they are aware of the presence of fake orders. Sales volume in our model affects customers only through product rankings. We note that many platforms do not explicitly display sales volume, although many do show the number of units in stock or the number of reviews. Future research can study a hypothetical scenario in which sales volume influences both customers’ search order and their evaluation of products, and how customers strategically discount the credibility of this signal in light of sales fakery.

The current model’s stylized ranking algorithm is purely based on the sales volume, and future research can incorporate more features such as product ratings, customer reviews, personal preferences, into the ranking algorithm. As stated earlier, sales volume may be correlated with, and thus can proxy some of these features. On a different note, we make the reasonable assumption that the platform does not know when customers arrive and thus cannot distinguish fake orders from real ones based on their timing. It might be deemed reckless of platforms to identify fake transactions on the sole basis of a flurry of orders being placed in a short time window.

Since brushing inflates the platform’s gross merchandise volume (GMV), an indicator often used by investors to gauge a platform’s financial performance, some worry that the platform may lack incentives to stem brushing (WSJ 2015a). In fact, there are allegations of JD.com giving merchants organizational support for brushing (Forbes 2015); relatedly, Alibaba’s reported revenue was probed by the U.S. Securities and Exchange Commission amid concerns of fictitious transactions (USA Today 2016). Future research can take the perspective of a platform and investigate how it balances customer welfare, GMV, and the capital market when regulating brushing.

Finally, while we set our paper in the context of e-commerce platforms to fix ideas, brushing, as a phenomenon, transcends e-commerce and finds its way in other business applications as well. Examples include film distributors buying up movie tickets to artificially boost a movie’s box-office rankings (Bloomberg 2017); podcasters pumping up subscription totals via fake accounts to manipulate the iTunes podcast charts (Business Insider 2018); pop music artists allegedly faking sales to rig the iTunes store rankings (Washington Post 2018); app developers inflating the number of downloads to be ranked higher on the leaderboard in the App Store and Google Play (Zhu et al. 2015). We hope that as the first analytical study of brushing, our research will draw more attention to this controversial yet pervasive practice and will inspire more future research on this topic.

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Appendix A: Proofs of the Main Results

Proof of Proposition 1

When product H is top-ranked, a focal customer's reservation utility for searching the second product, $\bar{U}_L$, is determined by

$$\bar{U}_L = -c + \gamma \max\{\bar{U}_L, v_L\} + (1 - \gamma) \max\{\bar{U}_L, v_0\}.$$

It is clear that $\bar{U}_L < v_L$, since otherwise, we have $\bar{U}_L = -c + \gamma \bar{U}_L + (1 - \gamma) \bar{U}_L = -c + \bar{U}_L$, a contradiction.

If $\bar{U}_L \leq v_0$, then we have $\bar{U}_L = -c + \gamma v_L + (1 - \gamma) v_0$. Therefore, $\bar{U}_L \leq v_0$ implies $c \geq \gamma(v_L - v_0)$. As the customer will search if and only if the realized value of the first product is less than the reservation utility, in this case $\bar{U}_L$, hence, the customer will never conduct search, since $\bar{U}_L \leq v_0$.

If $\bar{U}_L > v_0$, then we have $\bar{U}_L = -c + \gamma v_L + (1 - \gamma) \bar{U}_L \Rightarrow \bar{U}_L = v_L - c/\gamma$. Therefore, $\bar{U}_L > v_0$ implies $c < \gamma(v_L - v_0)$. In this case, the customer will search if and only if the realized value of the first product is v_0 .

When product L is top-ranked, customer's reservation utility for searching the second product, $\bar{U}_H$, is determined by

$$\bar{U}_H = -c + \gamma \max\{\bar{U}_H, v_H\} + (1 - \gamma) \max\{\bar{U}_H, v_0\}.$$

It is clear that $\bar{U}_H < v_H$, since otherwise, we have $\bar{U}_H = -c + \gamma \bar{U}_H + (1 - \gamma) \bar{U}_H = -c + \bar{U}_H$, a contradiction.

If $\bar{U}_H \leq v_0$, then we have $\bar{U}_H = -c + \gamma v_H + (1 - \gamma) v_0$. Therefore, $\bar{U}_H \leq v_0$ implies $c \geq \gamma(v_H - v_0)$. In this case, the customer will never conduct search.

If $\bar{U}_H > v_0$, then we have $\bar{U}_H = -c + \gamma v_H + (1 - \gamma) \bar{U}_H \Rightarrow \bar{U}_H = v_H - c/\gamma$. Therefore, $\bar{U}_H > v_0$ implies $c < \gamma(v_H - v_0)$. In this case, if the realized value of the first product is v_0 , then the customer will conduct search; otherwise, if the realized value of the first product is v_L , then the customer will conduct search if and only if $v_L < \bar{U}_H \Rightarrow c < \gamma(v_H - v_L)$.

The search strategies of customers are thus summarized in Table 2 of the paper.

Based on Table 2, we have:

Case 1. $v_H + v_0 \geq 2v_L$. In this case, we have $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$.

- If product H is top-ranked, then the customer will never conduct search, in which case, we have:

$$p_1 = 1, U_H = \gamma v_H + (1 - \gamma) v_0.$$

- If product L is top-ranked, then the customer will never conduct search, in which case, we have:

$$p_2 = 0, U_L = \gamma v_L + (1 - \gamma) v_0.$$

Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$.

- If product H is top-ranked, then the customer will never conduct search, in which case, we have:

$$p_1 = 1, U_H = \gamma v_H + (1 - \gamma) v_0.$$

• If product L is top-ranked, if the realized value of the first product is v_0 , then the customer will conduct search; if the realized value of the first product is v_L , then the customer will not conduct search, in which case, we have:

$$p_2 = (1 - \gamma) \left(\gamma + \frac{1 - \gamma}{2} \right) = \frac{1 - \gamma^2}{2}, \quad U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c].$$

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

• If product H is top-ranked, then the customer will never conduct search, in which case, we have:

$$p_1 = 1, \quad U_H = \gamma v_H + (1 - \gamma)v_0.$$

• If product L is top-ranked, then the customer will always conduct search, regardless of the realized value of the first product. Hence, we have:

$$p_2 = \gamma^2 + (1 - \gamma) \left(\gamma + \frac{1 - \gamma}{2} \right) = \frac{1 + \gamma^2}{2},$$

$$U_L = \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0] - c = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - c.$$

Case 1.4. $c < \gamma(v_L - v_0)$.

• If product H is top-ranked, then the customer will search if and only if the realized value of the first product is v_0 . Hence, we have:

$$p_1 = \gamma + \frac{(1 - \gamma)^2}{2} = \frac{1 + \gamma^2}{2}, \quad U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c].$$

• If product L is top-ranked, then the customer will always conduct search, regardless of the realized value of the first product. Hence, we have:

$$p_2 = \gamma^2 + (1 - \gamma) \left(\gamma + \frac{1 - \gamma}{2} \right) = \frac{1 + \gamma^2}{2},$$

$$U_L = \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0] - c = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - c. \quad \square$$

Proof of Corollary 1

Parts (i), (iii), (iv), (v) and (vi) of Corollary 1 are straightforward. We prove (ii).

In Case 1.1 of Proposition 1, it is obvious that $U_H > U_L$.

In Case 1.2 of Proposition 1, we have:

$$\begin{aligned} U_H - U_L &= \gamma^2 v_H + (1 - \gamma)\gamma v_0 - \gamma v_L + (1 - \gamma)c \\ &\geq \gamma[v_H - v_L - (1 - \gamma)(v_L - v_0)], \quad \because c \geq \gamma(v_H - v_L) \\ &> \gamma[v_L - v_0 - (1 - \gamma)(v_L - v_0)], \quad \because v_H + v_0 > 2v_L \\ &= \gamma^2(v_L - v_0) > 0. \end{aligned}$$

In Case 1.3 of Proposition 1, we have:

$$\begin{aligned} U_H - U_L &= -\gamma(1 - \gamma)v_L + \gamma(1 - \gamma)v_0 + c \\ &\geq -\gamma(1 - \gamma)v_L + \gamma(1 - \gamma)v_0 + v_H - v_L, \quad \because c \geq \gamma(v_H - v_L) \\ &= v_H - v_L - \gamma(1 - \gamma)(v_L - v_0), \quad \because v_H + v_0 > 2v_L \\ &\geq (v_L - v_0)[1 - \gamma(1 - \gamma)] > 0. \end{aligned}$$

In Case 1.4 of Proposition 1, we have $U_H - U_L = \gamma c > 0$. $\square$

Proof of Lemma 1

We first consider an auxiliary random walk $\hat{S}_t$ such that the transition probability is identical to S_t but the process stops if $\hat{S}_t \in \{T, -T\}$, where $T \in \mathbb{N}$, i.e., states $\{T, -T\}$ are absorbing states. Note that $S_t = \lim_{T \rightarrow \infty} \hat{S}_t$ almost surely. We define:

$$\hat{P}(s) \triangleq \mathbb{P} \left\{ \lim_{t \rightarrow \infty} \hat{S}_t = T \mid \hat{S}_0 = s \right\}.$$

By the Markov property, we have:

$$\begin{cases} \hat{P}(s) = p_1 \hat{P}(s+1) + (1-p_1) \hat{P}(s-1), & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \hat{P}(s) = p_2 \hat{P}(s+1) + (1-p_2) \hat{P}(s-1), & \text{if } s \in \{0\} \cup \mathbb{Z}^-. \end{cases} \quad (\text{A.1})$$

Note that the difference equations (A.1) are second order homogeneous equations. Hence, we have

$$\begin{cases} \hat{P}(s) = \hat{C}_1 \lambda_1^s + \hat{C}_2 \lambda_2^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \hat{P}(s) = \hat{D}_1 \lambda_3^s + \hat{D}_2 \lambda_4^s, & \text{if } s \in \{0\} \cup \mathbb{Z}^-, \end{cases}$$

where λ_1 and λ_2 are solutions to $\lambda^s = p_1 \lambda^{s+1} + (1-p_1) \lambda^{s-1}$; and λ_3 and λ_4 are solutions to $\lambda^s = p_2 \lambda^{s+1} + (1-p_2) \lambda^{s-1}$. We define $\lambda^{\bar{0}} = \lambda^0 \triangleq 1$ for all $\lambda \neq 0$ to be consistent with the convention $a^0 = 1, \forall a \neq 0$. The coefficients $\hat{C}_1, \hat{C}_2, \hat{D}_1$ and $\hat{D}_2$ can be derived from boundary conditions of $\hat{S}_t$.

After solving for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, we have

$$\begin{cases} \hat{P}(s) = \hat{C}_1 + \hat{C}_2 \left(\frac{1-p_1}{p_1} \right)^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \hat{P}(s) = \hat{D}_1 + \hat{D}_2 \left(\frac{1-p_2}{p_2} \right)^s, & \text{if } s \in \{0\} \cup \mathbb{Z}^-. \end{cases}$$

To solve for coefficients $\hat{C}_1, \hat{C}_2, \hat{D}_1, \hat{D}_2$, we use the following first-step conditioning and boundary conditions:

$$\begin{aligned} \begin{cases} \hat{P}(\bar{0}) = p_1 \hat{P}(1) + (1-p_1) \hat{P}(-1), \\ \hat{P}(0) = p_2 \hat{P}(1) + (1-p_2) \hat{P}(-1), \end{cases} & \Leftrightarrow \begin{cases} \hat{C}_1 + \hat{C}_2 = p_1 \left(\hat{C}_1 + \frac{1-p_1}{p_1} \hat{C}_2 \right) + (1-p_1) \left(\hat{D}_1 + \frac{p_2}{1-p_2} \hat{D}_2 \right), \\ \hat{D}_1 + \hat{D}_2 = p_2 \left(\hat{C}_1 + \frac{1-p_1}{p_1} \hat{C}_2 \right) + (1-p_2) \left(\hat{D}_1 + \frac{p_2}{1-p_2} \hat{D}_2 \right), \end{cases} \\ \begin{cases} \hat{P}(T) = 1 \Rightarrow \hat{C}_1 + \hat{C}_2 \left(\frac{1-p_1}{p_1} \right)^T = 1, \\ \hat{P}(-T) = 0 \Rightarrow \hat{D}_1 + \hat{D}_2 \left(\frac{p_2}{1-p_2} \right)^T = 0. \end{cases} \end{aligned}$$

This gives us:

$$\begin{cases} \hat{C}_1 = \frac{2p_1 + 2q_2 - 4p_1 q_2 + \alpha q_2 + 2p_1 q_2^2 + 2p_1^2 q_2 - p_1^2 - q_2^2 - \alpha q_2^2 + 2\alpha p_1 q_2^2 - 2p_1 q_2 \alpha - 1}{2p_1 + 2q_2 - 4p_1 q_2 + \beta p_1 + \alpha q_2 + 2p_1 q_2^2 + 2p_1^2 q_2 - p_1^2 - q_2^2 - \beta p_1^2 - \alpha q_2^2 + 2p_1^2 q_2 \beta + 2p_1 q_2^2 \alpha - 2p_1 q_2 \beta - 2p_1 q_2 \alpha - 1}, \\ \hat{C}_2 = \frac{p_1(2q_2 - 1)(p_1 - 1)}{2p_1 + 2q_2 - 4p_1 q_2 + \beta p_1 + \alpha q_2 + 2p_1 q_2^2 + 2p_1^2 q_2 - p_1^2 - q_2^2 - \beta p_1^2 - \alpha q_2^2 + 2p_1^2 q_2 \beta + 2p_1 q_2^2 \alpha - 2p_1 q_2 \beta - 2p_1 q_2 \alpha - 1}, \\ \hat{D}_1 = \frac{\alpha q_2(2p_1 - 1)(q_2 - 1)}{2p_1 + 2q_2 - 4p_1 q_2 + \beta p_1 + \alpha q_2 + 2p_1 q_2^2 + 2p_1^2 q_2 - p_1^2 - q_2^2 - \beta p_1^2 - \alpha q_2^2 + 2p_1^2 q_2 \beta + 2p_1 q_2^2 \alpha - 2p_1 q_2 \beta - 2p_1 q_2 \alpha - 1}, \\ \hat{D}_2 = -\frac{q_2(2p_1 - 1)(q_2 - 1)}{2p_1 + 2q_2 - 4p_1 q_2 + \beta p_1 + \alpha q_2 + 2p_1 q_2^2 + 2p_1^2 q_2 - p_1^2 - q_2^2 - \beta p_1^2 - \alpha q_2^2 + 2p_1^2 q_2 \beta + 2p_1 q_2^2 \alpha - 2p_1 q_2 \beta - 2p_1 q_2 \alpha - 1}, \end{cases}$$

where

$$q_2 \triangleq 1 - p_2 \text{ and } \alpha \triangleq \left(\frac{1 - q_2}{q_2} \right)^T, \quad \beta \triangleq \left(\frac{1 - p_1}{p_1} \right)^T.$$

Now we take T to infinity and prove (i) and (ii) of Lemma 1. We operate under the conditions: $p_1 + p_2 \geq 1$, $p_1 \geq p_2$ and $p_1 > 1/2$.

Part (i).

• When $p_2 > 1/2$, then we have $\lim_{T \rightarrow \infty} \alpha = +\infty$ and $\lim_{T \rightarrow \infty} \beta = 0$, which implies $C_1 = D_1 = 1$ and $C_2 = D_2 = 0$, where $C_i = \lim_{T \rightarrow \infty} \hat{C}_i$ and $D_i = \lim_{T \rightarrow \infty} \hat{D}_i$, with $i \in \{1, 2\}$. Hence, random walk S_t always drifts to $+\infty$ as t goes to infinity, regardless of initial state s .

• When $p_2 = 1/2$, then we have for $s \in \{\bar{0}, 1, 2, \dots\}$:

$$\begin{cases} C_1 = \lim_{T \rightarrow \infty} \lim_{p_2 \rightarrow 1/2} \hat{C}_1 = \lim_{T \rightarrow \infty} \frac{2Tp_1 - 2p_1 - T + 2p_1^2 + 1}{2Tp_1 - 2p_1 - T + 2p_1^2 + 1 - \left(\frac{1-p_1}{p_1}\right)^T (2p_1 - 2p_1^2)} = 1, \\ C_2 = \lim_{T \rightarrow \infty} \lim_{p_2 \rightarrow 1/2} \hat{C}_2 = \lim_{T \rightarrow \infty} \frac{2p_1(p_1 - 1)}{2Tp_1 - 2p_1 - T + 2p_1^2 + 1 - \left(\frac{1-p_1}{p_1}\right)^T (2p_1 - 2p_1^2)} = 0. \end{cases}$$

For $s \in \{\underline{0}, -1, -2, \dots\}$

$$P(s) = \lim_{T \rightarrow \infty} \lim_{p_2 \rightarrow 1/2} \left(D_1 + D_2 \left(\frac{1-p_2}{p_2} \right)^s \right) = \lim_{T \rightarrow \infty} \frac{(T+s)(2p_1 - 1)}{T(2p_1 - 1) - 2p_1 + 2p_1^2 + 1 + \left(\frac{1-p_1}{p_1}\right)^T (2p_1^2 - 2p_1)} = 1.$$

Hence, random walk S_t always drifts to $+\infty$ as t goes to infinity, regardless of initial state s when $p_2 \geq 1/2$.

Part (ii).

• When $p_2 < 1/2$, then we have $\lim_{T \rightarrow \infty} \beta = \lim_{T \rightarrow \infty} \alpha = 0$. Hence:

$$\begin{cases} C_1 = \lim_{T \rightarrow \infty} \hat{C}_1 = 1, \\ C_2 = \lim_{T \rightarrow \infty} \hat{C}_2 = \frac{p_1(2q_2 - 1)(p_1 - 1)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1}, \\ D_1 = \lim_{T \rightarrow \infty} \hat{D}_1 = 0, \\ D_2 = \lim_{T \rightarrow \infty} \hat{D}_2 = \frac{q_2(2p_1 - 1)(1 - q_2)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1}. \end{cases} \quad (\text{A.2})$$

Note that

$$2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1 = (2q_2 - 1)(1 - p_1)^2 + q_2^2(2p_1 - 1) > 0. \quad (\text{A.3})$$

And

$$p_1(2q_2 - 1)(1 - p_1) < (2q_2 - 1)(1 - p_1)^2 + q_2^2(2p_1 - 1) \Leftrightarrow -(2p_1 - 1)[(1 - q_2)^2 + p_1(2q_2 - 1)] < 0.$$

$$q_2(2p_1 - 1)(1 - q_2) < (2q_2 - 1)(1 - p_1)^2 + q_2^2(2p_1 - 1) \Leftrightarrow -(2q_2 - 1)[(1 - p_1)^2 + q_2(2p_1 - 1)] < 0.$$

This implies $C_2 \in (-1, 0]$ and $D_2 \in (0, 1)$. Also, we have

$$\begin{aligned} P(\bar{0}) > P(\underline{0}) &\Leftrightarrow 1 + C_2 > D_2 \\ &\Leftrightarrow 1 + \frac{p_1(2q_2 - 1)(p_1 - 1)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1} > \frac{q_2(2p_1 - 1)(1 - q_2)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1} \\ &\Leftrightarrow 1 > \frac{q_2(2p_1 - 1)(1 - q_2) - p_1(2q_2 - 1)(p_1 - 1)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1} \Leftrightarrow (2p_1 - 1)(2q_2 - 1)(p_1 - p_2) > 0, \end{aligned}$$

which is true by our assumption $p_1 + p_2 \geq 1$ and $p_1 > 1/2 > p_2$.

In addition, because

$$P(s) = \begin{cases} 1 + \frac{p_1(2q_2 - 1)(p_1 - 1)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1} \left(\frac{1-p_1}{p_1} \right)^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{P}^+, \\ \frac{q_2(2p_1 - 1)(1 - q_2)}{2p_1 + 2q_2 - 4p_1q_2 + 2p_1q_2^2 + 2p_1^2q_2 - p_1^2 - q_2^2 - 1} \left(\frac{1-p_2}{p_2} \right)^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-, \end{cases} \quad (\text{A.4})$$

it is easy to see that $P(s)$ increases in s , i.e., $\dots < P(-2) < P(-1) < P(\underline{0}) < P(\bar{0}) < P(1) < P(2) < \dots$ if $p_1 < 1$ and $\dots < P(-2) < P(-1) < P(\underline{0}) < P(\bar{0}) = P(s) \equiv 1$, $\forall s \in \{\bar{0}\} \cup \mathbb{Z}^+$ if $p_1 = 1$.

The derivation of $P'(s)$ is identical and is thus omitted. It is easy to verify that $P(s) + P'(s) = 1$ for all $s \in \mathcal{S}$. $\square$

Proof of Theorem 1

Under $v_H + v_0 \geq 2v_L$, $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$.

For $s = \bar{0}$, the random walk drifts to $+\infty$ with probability 1. For $s = \underline{0}$, the random walk drifts to $-\infty$ with probability 1. Hence, we have:

$$W_R = W_S = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)v_0].$$

Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$.

For $s = \bar{0}$, the random walk drifts to $+\infty$ with probability 1. For $s = \underline{0}$, the random walk drifts to $+\infty$ with probability

$$P(\underline{0}) = \frac{(1 - p_2)(2p_1 - 1)p_2}{2p_1 + 2(1 - p_2) - 4p_1(1 - p_2) + 2p_1(1 - p_2)^2 + 2p_1^2(1 - p_2) - p_1^2 - (1 - p_2)^2 - 1} = \frac{1 - \gamma^2}{1 + \gamma^2}.$$

Hence, we have:

$$\begin{aligned} W_R &= \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]], \\ W_S &= \frac{1}{2}\left[\gamma v_H + (1 - \gamma)v_0 + \frac{1 - \gamma^2}{1 + \gamma^2}[\gamma v_H + (1 - \gamma)v_0] + \frac{2\gamma^2}{1 + \gamma^2}\{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\right]. \end{aligned}$$

Note that

$$\begin{aligned} W_S > W_R &\Leftrightarrow \frac{1 - \gamma^2}{1 + \gamma^2}[\gamma v_H + (1 - \gamma)v_0] + \gamma v_L \frac{\gamma^2 - 1}{1 + \gamma^2} + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] \frac{\gamma^2 - 1}{\gamma^2 + 1} > 0 \\ &\Leftrightarrow \gamma v_H + (1 - \gamma)v_0 - \gamma v_L - (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] > 0 \\ &\Leftrightarrow \gamma^2 v_H + \gamma(1 - \gamma)v_0 - \gamma v_L + (1 - \gamma)c > 0, \end{aligned} \tag{A.5}$$

which is because

$$\begin{aligned} &\gamma^2 v_H + \gamma(1 - \gamma)v_0 - \gamma v_L + (1 - \gamma)c \geq \gamma[\gamma v_H + (1 - \gamma)v_0 - v_L + (1 - \gamma)(v_H - v_L)] \because c \geq \gamma(v_H - v_L) \\ &= \gamma[v_H - v_L + (1 - \gamma)(v_0 - v_L)] \geq \gamma^2(v_L - v_0) > 0 \because v_H + v_0 \geq 2v_L. \end{aligned}$$

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

For $s \in \{\bar{0}, \underline{0}\}$, the random walk drifts to $+\infty$ with probability 1. Hence, we have:

$$\begin{aligned} W_R &= \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c], \\ W_S &= \gamma v_H + (1 - \gamma)v_0. \end{aligned}$$

Note that

$$\begin{aligned} W_S - W_R &= \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 - \gamma[\gamma v_H + (1 - \gamma)v_L] - (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]] \\ &= \frac{1}{2}[(1 - \gamma)\gamma(v_0 - v_L) + c] > 0, \end{aligned}$$

which is because

$$(1 - \gamma)\gamma(v_0 - v_L) + c \geq (1 - \gamma)\gamma(v_0 - v_L) + \gamma(v_L - v_0) = \gamma^2(v_L - v_0) > 0.$$

Case 1.4. $c < \gamma(v_L - v_0)$.

For $s \in \{\bar{0}, \underline{0}\}$, the random walk drifts to $+\infty$ with probability 1. Hence, we have:

$$\begin{aligned} W_R &= \frac{1}{2} [\gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c] + \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0] - c], \\ W_S &= \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c]. \end{aligned}$$

Note that

$$\begin{aligned} W_S - W_R &= \frac{1}{2} [\gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c] - \gamma[\gamma v_H + (1 - \gamma)v_L] - (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0] + c] \\ &= c/2 > 0. \quad \square \end{aligned}$$

Proof of Lemma 2

Similar to the proof of Lemma 1, we consider the auxiliary random walk $\hat{S}_t$ such that the transition probability is identical to S_t but the process stops if $\hat{S}_t \in \{T, -T\}$, where $T \in \mathbb{N}$, i.e., states $\{T, -T\}$ are absorbing states. Note that $S_t = \lim_{T \rightarrow \infty} \hat{S}_t$ almost surely. Therefore, the corresponding value functions $\hat{V}_H(s)$ and $\hat{V}_L(s)$ are

$$\begin{cases} \hat{V}_H(s) = p(s)[R + \delta \hat{V}_H(s+1)] + [1 - p(s)]\delta \hat{V}_H(s-1), \\ \hat{V}_L(s) = p(s)\delta \hat{V}_L(s+1) + [1 - p(s)][R + \delta \hat{V}_L(s-1)], \end{cases} \quad \text{where } p(s) = \begin{cases} p_1, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ p_2, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-, \end{cases}$$

with boundary condition:

$$V_H(T) = V_H(-T) = V_L(T) = V_L(-T) = 0. \quad (\text{A.6})$$

Note that $V_i(s) = \lim_{T \rightarrow \infty} \hat{V}_i(s)$ almost surely, where $i \in \{H, L\}$.

The difference equations of $\hat{V}_i(s)$, where $i \in \{H, L\}$, are second order inhomogeneous equations. Hence, we have

$$\hat{V}_H(s) = \begin{cases} \frac{Rp_1}{1-\delta} + \hat{C}_1 r_1^s + \hat{C}_2 r_2^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \frac{Rp_2}{1-\delta} + \hat{A}_1 r_3^s + \hat{A}_2 r_4^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases} \quad \text{and } \hat{V}_L(s) = \begin{cases} \frac{Rq_1}{1-\delta} + \hat{D}_1 r_1^s + \hat{D}_2 r_2^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \frac{Rq_2}{1-\delta} + \hat{B}_1 r_3^s + \hat{B}_2 r_4^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-, \end{cases}$$

where $q_i \triangleq 1 - p_i$ with $i \in \{H, L\}$. We still define $a^{\bar{0}} = a^{\underline{0}} = 1$, $\forall a \neq 0$. $\frac{Rp_i}{1-\delta}$ is the solution to $y = p_i(R + \delta y) + (1 - p_i)\delta y$, $i \in \{H, L\}$. And $\frac{Rq_i}{1-\delta}$ is the solution to $y = p_i\delta y + (1 - p_i)(R + \delta y)$, $i \in \{H, L\}$. r_1, r_2 are solutions to $y = p_1\delta y^2 + (1 - p_1)\delta$ and r_3, r_4 are solutions to $y = p_2\delta y^2 + (1 - p_2)\delta$. This implies

$$\begin{aligned} r_1 &= \frac{1 + \sqrt{1 - 4\delta^2 p_1(1 - p_1)}}{2p_1\delta}, \quad \text{and } r_2 = \frac{1 - \sqrt{1 - 4\delta^2 p_1(1 - p_1)}}{2p_1\delta}, \\ r_3 &= \frac{1 + \sqrt{1 - 4\delta^2 p_2(1 - p_2)}}{2p_2\delta}, \quad \text{and } r_4 = \frac{1 - \sqrt{1 - 4\delta^2 p_2(1 - p_2)}}{2p_2\delta}. \end{aligned} \quad (\text{A.7})$$

Note that as $p_1, p_2, \delta \in [0, 1]$, we always have $1 - 4\delta^2 p_i(1 - p_i) \geq 0$, $\forall i \in \{1, 2\}$. And it is easy to verify that $r_2, r_4 \in [0, 1]$ and $r_1, r_3 \in [1, +\infty]$.

With the boundary conditions (A.6), we are able to determine the coefficients $\hat{A}_i, \hat{B}_i, \hat{C}, \hat{D}_i$ where $i \in \{1, 2\}$.

Plus, we have:

$$\begin{cases} \hat{V}_H(\bar{0}) = p_1[R + \delta \hat{V}_H(1)] + q_1\delta \hat{V}_H(-1), \\ \hat{V}_H(\underline{0}) = p_2[R + \delta \hat{V}_H(1)] + q_2\delta \hat{V}_H(-1), \end{cases} \quad \text{and } \begin{cases} \hat{V}_L(\bar{0}) = p_1\delta \hat{V}_L(1) + q_1[R + \delta \hat{V}_L(-1)], \\ \hat{V}_L(\underline{0}) = p_2\delta \hat{V}_L(1) + q_2[R + \delta \hat{V}_L(-1)]. \end{cases}$$

For $\hat{V}_H$, this leads to

$$\begin{aligned} & \begin{cases} \frac{Rp_1}{1-\delta} + \hat{C}_1 r_1^T + \hat{C}_2 r_2^T = 0, \\ \frac{Rp_2}{1-\delta} + \hat{A}_1(1/r_3)^T + \hat{A}_2(1/r_4)^T = 0, \\ \frac{Rp_1}{1-\delta} + \hat{C}_1 + \hat{C}_2 = p_1 \delta \left[\frac{R}{\delta} + \frac{Rp_1}{1-\delta} + \hat{C}_1 r_1 + \hat{C}_2 r_2 \right] + (1-p_1) \delta \left[\frac{Rp_2}{1-\delta} + \hat{A}_1(1/r_3) + \hat{A}_2(1/r_4) \right], \\ \frac{Rp_2}{1-\delta} + \hat{A}_1 + \hat{A}_2 = p_2 \delta \left[\frac{R}{\delta} + \frac{Rp_1}{1-\delta} + \hat{C}_1 r_1 + \hat{C}_2 r_2 \right] + (1-p_2) \delta \left[\frac{Rp_2}{1-\delta} + \hat{A}_1(1/r_3) + \hat{A}_2(1/r_4) \right]. \end{cases} \\ \Leftrightarrow & \begin{cases} \hat{C}_1 r_1^T + \hat{C}_2 r_2^T = -\frac{Rp_1}{1-\delta}, \\ \hat{A}_1(1/r_3)^T + \hat{A}_2(1/r_4)^T = -\frac{Rp_2}{1-\delta}, \\ (1-p_1)\delta(1/r_3)\hat{A}_1 + (1-p_1)\delta(1/r_4)\hat{A}_2 - (1-p_1\delta r_1)\hat{C}_1 - (1-p_1\delta r_2)\hat{C}_2 = \frac{\delta R(1-p_1)(p_1-p_2)}{1-\delta}, \\ [1-(1-p_2)\delta(1/r_3)]\hat{A}_1 + [1-(1-p_2)\delta(1/r_4)]\hat{A}_2 - p_2\delta r_1\hat{C}_1 - p_2\delta r_2\hat{C}_2 = \frac{R\delta p_2(p_1-p_2)}{1-\delta}. \end{cases} \end{aligned}$$

For $\hat{V}_L$, this leads to

$$\begin{aligned} & \begin{cases} \frac{R(1-p_1)}{1-\delta} + \hat{D}_1 r_1^T + \hat{D}_2 r_2^T = 0, \\ \frac{R(1-p_2)}{1-\delta} + \hat{B}_1(1/r_3)^T + \hat{B}_2(1/r_4)^T = 0, \\ \frac{R(1-p_1)}{1-\delta} + \hat{D}_1 + \hat{D}_2 = p_1 \delta \left[\frac{R(1-p_1)}{1-\delta} + \hat{D}_1 r_1 + \hat{D}_2 r_2 \right] + (1-p_1) \delta \left[\frac{R}{\delta} + \frac{R(1-p_2)}{1-\delta} + \hat{B}_1(1/r_3) + \hat{B}_2(1/r_4) \right], \\ \frac{R(1-p_2)}{1-\delta} + \hat{B}_1 + \hat{B}_2 = p_2 \delta \left[\frac{R(1-p_1)}{1-\delta} + \hat{D}_1 r_1 + \hat{D}_2 r_2 \right] + (1-p_2) \delta \left[\frac{R}{\delta} + \frac{R(1-p_2)}{1-\delta} + \hat{B}_1(1/r_3) + \hat{B}_2(1/r_4) \right]. \end{cases} \\ \Leftrightarrow & \begin{cases} \hat{D}_1 r_1^T + \hat{D}_2 r_2^T = -\frac{R(1-p_1)}{1-\delta}, \\ \hat{B}_1(1/r_3)^T + \hat{B}_2(1/r_4)^T = -\frac{R(1-p_2)}{1-\delta}, \\ (1-p_1)\delta(1/r_3)\hat{B}_1 + (1-p_1)\delta(1/r_4)\hat{B}_2 - (1-p_1\delta r_1)\hat{D}_1 - (1-p_1\delta r_2)\hat{D}_2 = \frac{R(1-p_1)\delta(p_2-p_1)}{1-\delta}, \\ [1-(1-p_2)\delta(1/r_3)]\hat{B}_1 + [1-(1-p_2)\delta(1/r_4)]\hat{B}_2 - p_2\delta r_1\hat{D}_1 - p_2\delta r_2\hat{D}_2 = \frac{(p_2-p_1)p_2\delta R}{1-\delta}. \end{cases} \end{aligned}$$

This gives:

$$\begin{aligned} \hat{A}_1 &= -\frac{(r_1 r_3)^T(u_1 v_2 + u_2 w_2) - (r_2 r_3)^T(u_1 v_1 + u_2 w_1) - k_1 r_3^T(v_1 w_2 - v_2 w_1) - k_2(r_1 r_3 r_4)^T(v_2 w_4 - v_4 w_2) + k_2(r_2 r_3 r_4)^T(v_1 w_4 - v_4 w_1)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\ \hat{A}_2 &= \frac{r_4^T[r_1^T(u_1 v_2 + u_2 w_2) - r_2^T(u_1 v_1 + u_2 w_1) - k_1(v_1 w_2 - v_2 w_1) - k_2(r_1 r_3)^T(v_2 w_3 - v_3 w_2) + k_2(r_2 r_3)^T(v_1 w_3 - v_3 w_1)]}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\ \hat{C}_1 &= \frac{(r_2 r_3)^T(u_1 v_3 + u_2 w_3) - (r_2 r_4)^T(u_1 v_4 + u_2 w_4) - k_1 r_3^T(v_2 w_3 - v_3 w_2) + k_1 r_4^T(v_2 w_4 - v_4 w_2) - k_2(r_2 r_3 r_4)^T(v_3 w_4 - v_4 w_3)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\ \hat{C}_2 &= -\frac{(r_1 r_3)^T(u_1 v_3 + u_2 w_3) - (r_1 r_4)^T(u_1 v_4 + u_2 w_4) - k_1 r_3^T(v_1 w_3 - v_3 w_1) + k_1 r_4^T(v_1 w_4 - v_4 w_1) - k_2(r_1 r_3 r_4)^T(v_3 w_4 - v_4 w_3)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)} \end{aligned}$$

$$\begin{aligned}
\hat{B}_1 &= \frac{(r_1 r_3)^T(u_1 v_2 + u_2 w_2) - (r_2 r_3)^T(u_1 v_1 + u_2 w_1) + k_3 r_3^T(v_1 w_2 - v_2 w_1) + k_4(r_1 r_3 r_4)^T(v_2 w_4 - v_4 w_2) - k_4(r_2 r_3 r_4)^T(v_1 w_4 - v_4 w_1)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\
\hat{B}_2 &= -\frac{r_4^T[r_1^T(u_1 v_2 + u_2 w_2) - r_2^T(u_1 v_1 + u_2 w_1) + k_3(v_1 w_2 - v_2 w_1) + k_4(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - k_4(r_2 r_3)^T(v_1 w_3 - v_3 w_1)]}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\
\hat{D}_1 &= -\frac{(r_2 r_3)^T(u_1 v_3 + u_2 w_3) - (r_2 r_4)^T(u_1 v_4 + u_2 w_4) + k_3 r_3^T(v_2 w_3 - v_3 w_2) - k_3 r_4^T(v_2 w_4 - v_4 w_2) + k_4(r_2 r_3 r_4)^T(v_3 w_4 - v_4 w_3)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)}, \\
\hat{D}_2 &= -\frac{(r_1 r_3)^T(u_1 v_3 + u_2 w_3) - (r_1 r_4)^T(u_1 v_4 + u_2 w_4) + k_3 r_3^T(v_1 w_3 - v_3 w_1) - k_3 r_4^T(v_1 w_4 - v_4 w_1) + k_4(r_1 r_3 r_4)^T(v_3 w_4 - v_4 w_3)}{(r_1 r_3)^T(v_2 w_3 - v_3 w_2) - (r_2 r_3)^T(v_1 w_3 - v_3 w_1) - (r_1 r_4)^T(v_2 w_4 - v_4 w_2) + (r_2 r_4)^T(v_1 w_4 - v_4 w_1)},
\end{aligned}$$

where

$$\begin{aligned}
w_1 &= 1 - \delta p_1 r_1, \quad w_2 = 1 - \delta p_1 r_2, \quad w_3 = (1 - p_1)\delta(1/r_3), \quad w_4 = (1 - p_1)\delta(1/r_4), \\
v_1 &= \delta p_2 r_1, \quad v_2 = \delta p_2 r_2, \quad v_3 = 1 - (1 - p_2)\delta(1/r_3), \quad v_4 = 1 - (1 - p_2)\delta(1/r_4), \\
u_1 &= p_1 R + \delta p_1 k_1 + (1 - p_1)\delta k_2 - k_1 = \frac{R\delta(p_2 - p_1)(1 - p_1)}{1 - \delta}, \\
u_2 &= p_2 R + \delta p_2 k_1 + (1 - p_2)\delta k_2 - k_2 = -\frac{R\delta p_2(p_2 - p_1)}{1 - \delta}, \\
k_1 &= \frac{Rp_1}{1 - \delta}, \quad k_2 = \frac{Rp_2}{1 - \delta}, \quad k_3 = \frac{R(1 - p_1)}{1 - \delta}, \quad k_4 = \frac{R(1 - p_2)}{1 - \delta}.
\end{aligned}$$

Note that

$$\begin{aligned}
A_1 &= \lim_{T \rightarrow \infty} \hat{A}_1, \\
&= -\lim_{T \rightarrow \infty} \frac{(u_1 v_2 + u_2 w_2) - (\frac{r_2}{r_1})^T(u_1 v_1 + u_2 w_1) - k_1 r_1^{-T}(v_1 w_2 - v_2 w_1) - k_2 r_4^T(v_2 w_4 - v_4 w_2) + k_2(\frac{r_2 r_4}{r_1})^T(v_1 w_4 - v_4 w_1)}{(v_2 w_3 - v_3 w_2) - (\frac{r_2}{r_1})^T(v_1 w_3 - v_3 w_1) - (\frac{r_4}{r_3})^T(v_2 w_4 - v_4 w_2) + (\frac{r_2 r_4}{r_1 r_3})^T(v_1 w_4 - v_4 w_1)}, \\
&= -\frac{u_1 v_2 + u_2 w_2}{v_2 w_3 - v_3 w_2} = -\frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]}, \\
A_2 &= \lim_{T \rightarrow \infty} \hat{A}_2, \\
&= \lim_{T \rightarrow \infty} \frac{r_4^T[r_3^{-T}(u_1 v_2 + u_2 w_2) - (\frac{r_2}{r_1 r_3})^T(u_1 v_1 + u_2 w_1) - k_1(r_1 r_3)^{-T}(v_1 w_2 - v_2 w_1) - k_2(v_2 w_3 - v_3 w_2) + k_2(\frac{r_2}{r_1})^T(v_1 w_3 - v_3 w_1)]}{(v_2 w_3 - v_3 w_2) - (\frac{r_2}{r_1})^T(v_1 w_3 - v_3 w_1) - (\frac{r_4}{r_3})^T(v_2 w_4 - v_4 w_2) + (\frac{r_2 r_4}{r_1 r_3})^T(v_1 w_4 - v_4 w_1)}, \\
&= 0, \\
C_1 &= \lim_{T \rightarrow \infty} \hat{C}_1, \\
&= \lim_{T \rightarrow \infty} \frac{(\frac{r_2}{r_1})^T(u_1 v_3 + u_2 w_3) - (\frac{r_2 r_4}{r_1 r_3})^T(u_1 v_4 + u_2 w_4) - k_1 r_1^{-T}(v_2 w_3 - v_3 w_2) + k_1(\frac{r_4}{r_1 r_3})^T(v_2 w_4 - v_4 w_2) - k_2(\frac{r_2 r_4}{r_1})^T(v_3 w_4 - v_4 w_3)}{(v_2 w_3 - v_3 w_2) - (\frac{r_2}{r_1})^T(v_1 w_3 - v_3 w_1) - (\frac{r_4}{r_3})^T(v_2 w_4 - v_4 w_2) + (\frac{r_2 r_4}{r_1 r_3})^T(v_1 w_4 - v_4 w_1)},
\end{aligned}$$

$$\begin{aligned}
&= 0, \\
C_2 &= \lim_{T \rightarrow \infty} \hat{C}_2, \\
&\quad (u_1 v_3 + u_2 w_3) - \left(\frac{r_4}{r_3}\right)^T (u_1 v_4 + u_2 w_4) - k_1 r_1^{-T} (v_1 w_3 - v_3 w_1) \\
&\quad + k_1 \left(\frac{r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1) - k_2 r_4^T (v_3 w_4 - v_4 w_3) \\
&= - \lim_{T \rightarrow \infty} \frac{(v_2 w_3 - v_3 w_2) - \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1) - \left(\frac{r_4}{r_3}\right)^T (v_2 w_4 - v_4 w_2) + \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1)}{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}, \\
&= - \frac{u_1 v_3 + u_2 w_3}{v_2 w_3 - v_3 w_2} = \frac{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}{(1 - \delta) [(\delta p_1 r_2 - 1) (\delta (p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)]}, \\
B_1 &= \lim_{T \rightarrow \infty} \hat{B}_1, \\
&\quad (u_1 v_2 + u_2 w_2) - \left(\frac{r_2}{r_1}\right)^T (u_1 v_1 + u_2 w_1) + k_3 r_1^{-T} (v_1 w_2 - v_2 w_1) \\
&\quad + k_4 r_4^T (v_2 w_4 - v_4 w_2) - k_4 \left(\frac{r_2 r_4}{r_1}\right)^T (v_1 w_4 - v_4 w_1) \\
&= \lim_{T \rightarrow \infty} \frac{(v_2 w_3 - v_3 w_2) - \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1) - \left(\frac{r_4}{r_3}\right)^T (v_2 w_4 - v_4 w_2) + \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1)}{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}, \\
&= \frac{u_1 v_2 + u_2 w_2}{v_2 w_3 - v_3 w_2} = -A_1, \\
B_2 &= \lim_{T \rightarrow \infty} \hat{B}_2, \\
&\quad r_4^T [r_3^{-T} (u_1 v_2 + u_2 w_2) - \left(\frac{r_2}{r_1 r_3}\right)^T (u_1 v_1 + u_2 w_1) + k_3 (r_1 r_3)^{-T} (v_1 w_2 - v_2 w_1) \\
&\quad + k_4 (v_2 w_3 - v_3 w_2) - k_4 \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1)] \\
&= - \lim_{T \rightarrow \infty} \frac{(v_2 w_3 - v_3 w_2) - \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1) - \left(\frac{r_4}{r_3}\right)^T (v_2 w_4 - v_4 w_2) + \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1)}{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}, \\
&= 0, \\
D_1 &= \lim_{T \rightarrow \infty} \hat{D}_1, \\
&\quad \left(\frac{r_2}{r_1}\right)^T (u_1 v_3 + u_2 w_3) - \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (u_1 v_4 + u_2 w_4) + k_3 r_1^{-T} (v_2 w_3 - v_3 w_2) \\
&\quad - k_3 \left(\frac{r_4}{r_1 r_3}\right)^T (v_2 w_4 - v_4 w_2) + k_4 \left(\frac{r_2 r_4}{r_1}\right)^T (v_3 w_4 - v_4 w_3) \\
&= - \lim_{T \rightarrow \infty} \frac{(v_2 w_3 - v_3 w_2) - \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1) - \left(\frac{r_4}{r_3}\right)^T (v_2 w_4 - v_4 w_2) + \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1)}{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}, \\
&= 0, \\
D_2 &= \lim_{T \rightarrow \infty} \hat{D}_2, \\
&\quad (u_1 v_3 + u_2 w_3) - \left(\frac{r_4}{r_3}\right)^T (u_1 v_4 + u_2 w_4) + k_3 r_1^{-T} (v_1 w_3 - v_3 w_1) \\
&\quad - k_3 \left(\frac{r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1) + k_4 r_4^T (v_3 w_4 - v_4 w_3) \\
&= \lim_{T \rightarrow \infty} \frac{(v_2 w_3 - v_3 w_2) - \left(\frac{r_2}{r_1}\right)^T (v_1 w_3 - v_3 w_1) - \left(\frac{r_4}{r_3}\right)^T (v_2 w_4 - v_4 w_2) + \left(\frac{r_2 r_4}{r_1 r_3}\right)^T (v_1 w_4 - v_4 w_1)}{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}, \\
&= \frac{u_1 v_3 + u_2 w_3}{v_2 w_3 - v_3 w_2} = -C_2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
V_H(s) &= \begin{cases} \frac{R p_1}{1 - \delta} + \frac{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}{(1 - \delta) [(\delta p_1 r_2 - 1) (\delta (p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)]} r_2^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \frac{R p_2}{1 - \delta} - \frac{R \delta p_2 r_3 (p_1 - p_2) (1 - \delta r_2)}{(1 - \delta) [(\delta p_1 r_2 - 1) (\delta (p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)]} r_3^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases} \\
V_L(s) &= \begin{cases} \frac{R (1 - p_1)}{1 - \delta} - \frac{R \delta (p_1 - p_2) (1 - p_1) (r_3 - \delta)}{(1 - \delta) [(\delta p_1 r_2 - 1) (\delta (p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)]} r_2^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+, \\ \frac{R (1 - p_2)}{1 - \delta} + \frac{R \delta p_2 r_3 (p_1 - p_2) (1 - \delta r_2)}{(1 - \delta) [(\delta p_1 r_2 - 1) (\delta (p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)]} r_3^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases} \quad (\text{A.8})
\end{aligned}$$

Now we proceed to prove (i) and (ii) of this lemma.

Part (i).

Note that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) = (\delta - r_3)(1 - \delta p_1 r_2) - \delta p_2(1 - \delta r_2) < (\delta - r_3)(1 - \delta p_1 r_2) < 0. \quad (\text{A.9})$$

And

$$V_H(\bar{0}) - V_H(\underline{0}) = \frac{R(p_1 - p_2)}{1 - \delta} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) = V_L(\underline{0}) - V_L(\bar{0}).$$

Note that

$$(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2) \leq 0 \Leftrightarrow \frac{1 - p_1}{1 - \delta r_2} \leq \frac{p_2}{1 - \delta/r_3}. \quad (\text{A.10})$$

Note that $\frac{1 - p_1}{1 - \delta r_2} \leq \frac{p_2}{1 - \delta/r_3}$ is a special case of $\frac{(1 - p_1)(1 - r_2^Q)}{1 - \delta r_2} \leq \frac{p_2(1 - r_3^{-Q})}{1 - \delta/r_3}$ when $Q = 0$. Here we prove the general case, i.e., $Q \geq 0$.

We define:

$$F_1(p) \triangleq \frac{(1 - p) \left[1 - \left(\frac{1 - \sqrt{1 - 4\delta^2 p(1 - p)}}{2p\delta} \right)^Q \right]}{1 - \delta \left(\frac{1 - \sqrt{1 - 4\delta^2 p(1 - p)}}{2p\delta} \right)} \text{ and } F_2(p) \triangleq \frac{p \left[1 - \left(\frac{1 - \sqrt{1 - 4\delta^2 p(1 - p)}}{2\delta(1 - p)} \right)^Q \right]}{1 - \delta \left(\frac{1 - \sqrt{1 - 4\delta^2 p(1 - p)}}{2\delta(1 - p)} \right)}. \quad (\text{A.11})$$

It is easy to verify that $F_1(1 - p) = F_2(p)$ for $p \in [0, 1]$ and hence, $F_1(p)$ and $F_2(p)$ are symmetric with respect to $p = 1/2$. Hence,

$$\frac{(1 - p_1)(1 - r_2^Q)}{1 - \delta r_2} \leq \frac{p_2(1 - r_3^{-Q})}{1 - \delta/r_3} \Leftrightarrow p_1 + p_2 \geq 1, \quad (\text{A.12})$$

which is ensured by (iv) of Corollary 1. Hence, we have $V_H(\bar{0}) \geq V_H(\underline{0})$ and $V_L(\underline{0}) \geq V_L(\bar{0})$.

Hence, $V_H(s)$ increases in s (i.e., $\dots \leq V_H(-2) \leq V_H(-1) \leq V_H(\underline{0}) \leq V_H(\bar{0}) \leq V_H(1) \leq V_H(2) \leq \dots$). $V_L(s)$ decreases in s (i.e., $\dots \geq V_L(-2) \geq V_L(-1) \geq V_L(\underline{0}) \geq V_L(\bar{0}) \geq V_L(1) \geq V_L(2) \geq \dots$).

Part (ii).

Under $\beta_H = \beta_L = 0$, the marginal gain of seller L if he brushes is:

$$\begin{aligned} M_L &= \frac{R(1 - p_2)}{1 - \delta} + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_3^{-Q} \\ &\quad - \frac{1}{2} \left(\frac{R(2 - p_1 - p_2)}{1 - \delta} - \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) \\ &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

The marginal gain of seller H if he brushes is

$$\begin{aligned} M_H &= \frac{Rp_1}{1 - \delta} + \frac{R\delta(p_1 - p_2)(1 - p_1)(r_3 - \delta)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_2^Q \\ &\quad - \frac{1}{2} \left(\frac{R(p_1 + p_2)}{1 - \delta} + \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) \\ &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Then we have:

$$M_L - M_H = \frac{R(p_1 - p_2)}{2(1 - \delta)} \frac{2\delta[(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \geq 0,$$

since

$$(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1) \leq 0 \Leftrightarrow \frac{(1 - p_1)(1 - r_2^Q)}{1 - \delta r_2} \leq \frac{p_2(1 - r_3^{-Q})}{1 - \delta/r_3} \Leftrightarrow p_1 + p_2 \geq 1,$$

based on (A.12).

Under $\beta_H = \beta_L = Q$, the marginal loss of seller L if he does not brush is:

$$\begin{aligned} M'_L &= -\frac{R(1 - p_1)}{1 - \delta} + \frac{R\delta(p_1 - p_2)(1 - p_1)(r_3 - \delta)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_2^Q \\ &\quad + \frac{1}{2} \left(\frac{R(2 - p_1 - p_2)}{1 - \delta} - \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) \\ &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

The marginal loss of seller H if he does not brush is:

$$\begin{aligned} M'_H &= -\frac{Rp_2}{1 - \delta} + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_3^{-Q} \\ &\quad + \frac{1}{2} \left(\frac{R(p_1 + p_2)}{1 - \delta} + \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) \\ &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Then we have:

$$M'_H - M'_L = \frac{R(p_1 - p_2)}{2(1 - \delta)} \frac{2\delta[(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} = M_L - M_H \geq 0. \quad \square$$

Proof of Proposition 2

Under $p_1 + p_2 \geq 1$, $p_1 \geq p_2$ and $p_1 > 1/2$, based on $V_H(s)$ and $V_L(s)$, we have:

Part (i).

Assuming $(\beta_H^*, \beta_L^*) = (1, 1)$ is an equilibrium, then for H seller we have:

$$\begin{aligned} \pi_H(\beta_H = 1, \beta_L = 1) &\geq \pi_H(\beta_H = 0, \beta_L = 1) \\ \Leftrightarrow \frac{1}{2} \left(\frac{R(p_1 + p_2)}{1 - \delta} + \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) &- c_B Q \\ \geq \frac{Rp_2}{1 - \delta} - \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_3^{-Q} \\ \Leftrightarrow c_B Q &\leq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right), \end{aligned}$$

where r_2, r_3 are defined in (A.7).

Note that if $p_1 > p_2$, then

$$\begin{aligned} \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) &> 0 \\ \Leftrightarrow 1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} &> 0 \\ \Leftrightarrow (\delta - r_3)(1 - \delta p_1 r_2) - \delta p_2(1 - \delta r_2) + \delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)] &< 0 \\ \Leftrightarrow (\delta - r_3)[1 - \delta + \delta p_1(1 - r_2)] - \delta p_2(1 - \delta r_2)[1 + r_3(1 - 2r_3^{-Q})] &< 0, \end{aligned} \tag{A.13}$$

which is obviously true (set $Q = 1$).

For seller L we have:

$$\begin{aligned}
\pi_L(\beta_H = 1, \beta_L = 1) &\geq \pi_L(\beta_H = 1, \beta_L = 0) \\
&\Leftrightarrow \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) - c_BQ \\
&\geq \frac{R(1-p_1)}{1-\delta} - \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_2^Q \\
&\Leftrightarrow c_BQ \leq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

Note that if $p_1 > p_2$, then

$$\begin{aligned}
&\frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right) > 0 \\
&\Leftrightarrow 1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} > 0 \\
&\Leftrightarrow (\delta-r_3)(1-\delta p_1r_2) - \delta p_2(1-\delta r_2) + \delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)] \\
&\Leftrightarrow (\delta-r_3)[(1-\delta p_1r_2) + \delta(1-p_1)(1-2r_2^Q)] - \delta p_2(1-\delta r_2)(1-r_3) < 0.
\end{aligned} \tag{A.14}$$

The term $(1-\delta p_1r_2) + \delta(1-p_1)(1-2r_2^Q)$ decreases in r_2 and when $r_2 = 1$, this term is $1-\delta > 0$.

Note that

$$\begin{aligned}
&\frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right) \\
&\geq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right) \\
&\Leftrightarrow (1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1) \leq (1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2) \\
&\Leftrightarrow (1-p_1)(r_3-\delta)(1-r_2^Q) + p_2r_3(1-\delta r_2)(r_3^{-Q}-1) \leq 0 \\
&\Leftrightarrow \frac{(1-p_1)(1-r_2^Q)}{1-\delta r_3} \leq \frac{p_2(1-r_3^{-Q})}{1-\delta/r_3} \Leftrightarrow p_1 + p_2 \geq 1,
\end{aligned}$$

which is based on (A.12). Hence, when

$$c_B < \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right),$$

$(\beta_H^*, \beta_L^*) = (1, 1)$ is a pure-strategy equilibrium. The uniqueness will be proved later.

Part (ii).

Assuming $(\beta_H^*, \beta_L^*) = (0, 0)$ is an equilibrium, then for seller H , we have:

$$\begin{aligned}
\pi_H(\beta_H = 0, \beta_L = 0) &\geq \pi_H(\beta_H = 1, \beta_L = 0) \\
&\Leftrightarrow \frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) \\
&\geq \frac{Rp_1}{1-\delta} + \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_2^Q - c_BQ \\
&\Leftrightarrow c_BQ \geq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

For seller L , we have:

$$\begin{aligned}
\pi_L(\beta_H=0, \beta_L=0) &\geq \pi_L(\beta_H=0, \beta_L=1) \\
&\Leftrightarrow \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) \\
&\geq \frac{R(1-p_2)}{1-\delta} + \frac{R\delta p_2r_3(p_1-p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_3^{-Q} - c_BQ \\
&\Leftrightarrow c_BQ \geq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

Hence, when

$$c_B > \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right)$$

$(\beta^*=0, \beta^*=0)$ is a pure-strategy equilibrium and the uniqueness will be proved in the subsequent analysis.

We show that $(\beta_H^*=1, \beta_L^*=0)$ cannot be a pure equilibrium except when

$$c_B = \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right). \quad (\text{A.15})$$

This is because assuming $(\beta_H^*=1, \beta_L^*=0)$ is an equilibrium, then for seller H , we have:

$$\begin{aligned}
\pi_H(\beta_H=1, \beta_L=0) &\geq \pi_H(\beta_H=0, \beta_L=0) \\
&\Leftrightarrow \frac{Rp_1}{1-\delta} + \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_2^Q - c_BQ \\
&\geq \frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) \\
&\Leftrightarrow c_BQ \leq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

For seller L , we have:

$$\begin{aligned}
\pi_L(\beta_H=1, \beta_L=0) &\geq \pi_L(\beta_H=1, \beta_L=1) \\
&\Leftrightarrow \frac{R(1-p_1)}{1-\delta} - \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_2^Q \\
&\geq \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) - c_BQ \\
&\Leftrightarrow c_BQ \geq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^Q-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

Hence, if (A.15) does not hold, then we clearly have a contradiction.

Similarly, $(\beta_H^*=0, \beta_L^*=1)$ cannot be a pure equilibrium except when

$$c_B = \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right). \quad (\text{A.16})$$

This is because assuming $(\beta_H^*=0, \beta_L^*=1)$ is an equilibrium, then for seller H , we have:

$$\begin{aligned}
\pi_H(\beta_H=0, \beta_L=1) &\geq \pi_H(\beta_H=1, \beta_L=1) \\
&\Leftrightarrow \frac{Rp_2}{1-\delta} - \frac{R\delta p_2r_3(p_1-p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} r_3^{-Q} \\
&\geq \frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)]} \right) - c_BQ \\
&\Leftrightarrow c_BQ \geq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).
\end{aligned}$$

For seller L , we have:

$$\begin{aligned}
\pi_L(\beta_H = 0, \beta_L = 1) &\geq \pi_H(\beta_H = 0, \beta_L = 0) \\
&\Leftrightarrow \frac{R(1-p_2)}{1-\delta} + \frac{R\delta p_2 r_3 (p_1 - p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_3^{-Q} - c_B Q \\
&\geq \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right) \\
&\Leftrightarrow c_B Q \leq \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2 r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right).
\end{aligned}$$

Hence, if (A.16) does not hold, then we clearly have a contradiction.

Therefore, the uniqueness of the pure strategy equilibrium $(\beta_H^* = 1, \beta_L^* = 1)$ when $C < \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_3^Q-1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right)$ is proved. In addition, it is obvious that $(\beta_H^* = 1, \beta_L^* = 1)$ is a dominant strategy equilibrium in this cases.

Similarly, the uniqueness of the pure strategy equilibrium $(\beta_H^* = 0, \beta_L^* = 0)$ when $C > \frac{R(p_1-p_2)}{2Q(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta) + p_2 r_3(1-\delta r_2)(2r_3^{-Q}-1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right)$ is proved. In addition, it is obvious that $(\beta_H^* = 0, \beta_L^* = 0)$ is a dominant strategy equilibrium in this cases.

Part (iii).

Now we consider mixed strategy.

Seller H 's expected utility when he brushes is:

$$\begin{aligned}
\beta_L &\left(\frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right) - c_B Q \right) \\
&+ (1-\beta_L) \left(\frac{R p_1}{1-\delta} + \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_2^Q - c_B Q \right).
\end{aligned}$$

Seller H 's expected utility when he does not brush is:

$$\begin{aligned}
\beta_L &\left(\frac{R p_2}{1-\delta} - \frac{R\delta p_2 r_3 (p_1 - p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_3^{-Q} \right) \\
&+ (1-\beta_L) \frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right).
\end{aligned}$$

Equate the above two expressions, we have:

$$\beta_L = \frac{c_B Q - \frac{R(p_1-p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_3^Q-1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right)}{\frac{R\delta(p_1-p_2)}{1-\delta} \frac{(1-p_1)(r_3-\delta)(1-r_2^Q) + p_2 r_3(1-\delta r_2)(r_3^{-Q}-1)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)}}.$$

Seller L 's expected utility when she brushes is:

$$\begin{aligned}
\beta_H &\left(\frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right) - c_B Q \right) \\
&+ (1-\beta_H) \left(\frac{R(1-p_2)}{1-\delta} + \frac{R\delta p_2 r_3 (p_1 - p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_3^{-Q} - c_B Q \right).
\end{aligned}$$

Seller L 's expected utility when she does not brush is:

$$\begin{aligned}
\beta_H &\left(\frac{R(1-p_1)}{1-\delta} - \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_2^Q \right) \\
&+ (1-\beta_H) \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right).
\end{aligned}$$

Equate the above two expressions, we have:

$$\beta_H = \frac{c_B Q - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right)}{\frac{R\delta(p_2 - p_1)}{1 - \delta} \frac{(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}}.$$

Note that as $p_1 + p_2 \geq 1$, we have $(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1) \leq 0$ based on (A.12). Hence

$$\begin{aligned} \beta_H \in (0, 1) \text{ and } \beta_L \in (0, 1) &\Leftrightarrow \\ \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) &< c_B Q \\ &< \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right), \end{aligned}$$

We also have:

$$\beta_H > \beta_L \Leftrightarrow c_B Q < \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[r_2^Q(1 - p_1)(r_3 - \delta) + p_2 r_3^{1-Q}(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right)$$

It is easy to verify that

$$\begin{aligned} &\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &< \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[r_2^Q(1 - p_1)(r_3 - \delta) + p_2 r_3^{1-Q}(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &< \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right), \end{aligned}$$

We define

$$\begin{aligned} \bar{C} &\triangleq \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right), \\ \underline{C} &\triangleq \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right), \\ \tilde{C} &\triangleq \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 + \frac{\delta[r_2^Q(1 - p_1)(r_3 - \delta) + p_2 r_3^{1-Q}(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned} \tag{A.17}$$

Note that $\beta_H > \beta_L$ if and only if $\underline{C} \leq c_B < \tilde{C}$.

Part (iv) and (v).

It is clear that when $c_B = \underline{C}$, there are two pure-strategy equilibria: $(\beta_H^*, \beta_L^*) = (1, 1)$ and $(\beta_H^*, \beta_L^*) = (1, 0)$.

In this case, we have

$$\begin{aligned} \pi_L(\beta_H^* = 1, \beta_L^* = 1) &= \frac{1}{2} \left(\frac{R(2 - p_1 - p_2)}{1 - \delta} - \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right) - c_B Q, \\ \pi_L(\beta_H^* = 1, \beta_L^* = 0) &= \frac{R(1 - p_1)}{1 - \delta} - \frac{R\delta(p_1 - p_2)(1 - p_1)(r_3 - \delta)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_2^Q. \end{aligned}$$

It is easy to verify that $\pi_L(\beta_H^* = 1, \beta_L^* = 1) = \pi_L(\beta_H^* = 1, \beta_L^* = 0)$ when $c_B = \underline{C}$. As $\beta_H^* = 1$ is the dominant strategy for $c_B \leq \underline{C}$, the set of all equilibria when $c_B = \underline{C}$ is: (β_H^*, β_L^*) with $\beta_H^* = 1$ and $\beta_L^* \in [0, 1]$.

When $c_B = \bar{C}$, there are two pure-strategy equilibria: $(\beta_H^*, \beta_L^*) = (0, 0)$ and $(\beta_H^*, \beta_L^*) = (0, 1)$. In this case, we have

$$\begin{aligned} \pi_L(\beta_H^* = 0, \beta_L^* = 0) &= \frac{1}{2} \left(\frac{R(2 - p_1 - p_2)}{1 - \delta} - \frac{R\delta(p_1 - p_2)[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right), \\ \pi_L(\beta_H^* = 0, \beta_L^* = 1) &= \frac{R(1 - p_2)}{1 - \delta} + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_3^{-Q} - c_B Q. \end{aligned}$$

It is easy to verify that $\pi_L(\beta_H^* = 0, \beta_L^* = 0) = \pi_L(\beta_H^* = 0, \beta_L^* = 0)$ when $c_B = \bar{C}$. As $\beta_H^* = 0$ is the dominant strategy for $c_B \geq \bar{C}$, the set of all equilibria when $c_B = \bar{C}$ is: (β_H^*, β_L^*) with $\beta_H^* = 0$ and $\beta_L^* \in [0, 1]$.

Now we show $\bar{C}$ and $\underline{C}$ are increasing in c . First, by (A.13) and (A.14), it is obvious that $\bar{C} = \underline{C} = 0$ if and only if $p_1 = p_2$. In addition, by Proposition 1, we know that $p_1 = p_2$ if and only if $c \leq \gamma(v_L - v_0)$. Therefore, $\bar{C} = \underline{C} = 0$ if and only if $c \leq \gamma(v_L - v_0)$.

Based on Proposition 1, we have :

Case 1. $v_H + v_0 \geq 2v_L$. In this case, we have $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$. In this case, we have $p_1 = 1$ and $p_2 = 0$, which implies $r_2 = 0$ and $r_3 = \infty$. Hence, we have:

$$\begin{aligned}\bar{C}(c) &= \lim_{p_2 \rightarrow 0} \frac{R(1-p_2)}{2(1-\delta)Q} \left(1 + \frac{\delta p_2(2r_3^{-Q} - 1)}{-(\delta(p_2 - 1)/r_3 + 1)} \right) = \frac{R}{2(1-\delta)Q}, \\ \underline{C}(c) &= \lim_{p_2 \rightarrow 0} \frac{R(1-p_2)}{2(1-\delta)Q} \left(1 + \frac{\delta p_2}{-(\delta(p_2 - 1)/r_3 + 1)} \right) = \frac{R}{2(1-\delta)Q}.\end{aligned}$$

Hence, when $c \geq \gamma(v_H - v_0)$, we have $\bar{C} = \underline{C} > 0$.

Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$. In this case, we have $p_1 = 1$ and $p_2 = \frac{1-\gamma^2}{2}$, which implies $r_2 = 0$ and $r_3 = \frac{1+\sqrt{\delta^2\gamma^4-\delta^2+1}}{\delta(1-\gamma^2)}$. Hence, we have:

$$\begin{aligned}\bar{C}(c) &= \frac{R(1-p_2)}{2(1-\delta)Q} \left(1 - \frac{\delta p_2(2r_3^{-Q} - 1)}{\delta(p_2 - 1)/r_3 + 1} \right) = \frac{R}{2(1-\delta)Q} \left(1 - p_2 - \frac{1 - \sqrt{1 - 4\delta^2 p_2(1-p_2)}}{2\delta} (2r_3^{-Q} - 1) \right), \\ \underline{C}(c) &= \frac{R(1-p_2)}{2(1-\delta)Q} \left(1 - \frac{\delta p_2}{\delta(p_2 - 1)/r_3 + 1} \right) = \frac{R}{2(1-\delta)Q} \left(1 - p_2 - \frac{1 - \sqrt{1 - 4\delta^2 p_2(1-p_2)}}{2\delta} \right).\end{aligned}\tag{A.18}$$

It is clear that $\bar{C}(c) > \underline{C}(c)$ for $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$. This is because $2r_3^{-Q} - 1 \leq 1 \Leftrightarrow r_3^{-Q} < 1$, since $r_3 > 1$.

It is clear that for $c_1 \geq \gamma(v_H - v_0)$ and $\gamma(v_H - v_L) \leq c_2 < \gamma(v_H - v_0)$, then $\underline{C}(c_1) > \underline{C}(c_2)$, as

$$1 - p_2 - \frac{1 - \sqrt{1 - 4\delta^2 p_2(1-p_2)}}{2\delta} \leq 1.$$

It is also clear that $\bar{C}(c_1) > \bar{C}(c_2)$, as

$$1 - p_2 - \frac{1 - \sqrt{1 - 4\delta^2 p_2(1-p_2)}}{2\delta} (2r_3^{-Q} - 1) \leq 1 - p_2 + \frac{1 - \sqrt{1 - 4\delta^2 p_2(1-p_2)}}{2\delta} \leq 1,$$

which is equivalent to $1 - 2\delta p_2 \leq \sqrt{1 - 4\delta^2 p_2(1-p_2)}$. When $p_2 \geq 1/(2\delta)$, the inequality holds trivially; when $p_2 < 1/(2\delta)$, then $1 - 2\delta p_2 \leq \sqrt{1 - 4\delta^2 p_2(1-p_2)} \Leftrightarrow \delta < 1$.

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$. In this case, we have $p_1 = 1$ and $p_2 = \frac{1+\gamma^2}{2}$, which implies $r_2 = 0$ and $r_3 = \frac{1+\sqrt{\delta^2\gamma^4-\delta^2+1}}{\delta(1+\gamma^2)}$. The expression of $\bar{C}(c)$ and $\underline{C}(c)$ are the same as those in Case 1.2. Therefore, $\bar{C}(c) > \underline{C}(c)$ for $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

Note that for $\gamma(v_H - v_L) \leq c_2 < \gamma(v_H - v_0)$, $p_2(c_2) = \frac{1-\gamma^2}{2}$ and for $\gamma(v_L - v_0) \leq c_3 < \gamma(v_H - v_L)$, $p_2(c_3) = \frac{1+\gamma^2}{2}$. By plugging $p_2(c_2)$ and $p_2(c_3)$ into (A.18), it is clear that $\bar{C}(c_2) > \bar{C}(c_3)$ and $\underline{C}(c_2) > \underline{C}(c_3)$.

Case 1.4. $c < \gamma(v_L - v_0)$. In this case, we have $p_1 = p_2 = \frac{1+\gamma^2}{2}$, which implies $\bar{C}(c_4) = \underline{C}(c_4) = 0$, where $c_4 < \gamma(v_L - v_0)$. Note that we always have $\bar{C} \geq 0$ and $\underline{C} \geq 0$. Hence, $\bar{C}(c_3) \geq \bar{C}(c_4)$ and $\underline{C}(c_3) \geq \underline{C}(c_4)$. $\square$

Proof of Theorem 2

Part (i).

Since $v_H + v_0 \geq 2v_L$, we have $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$.

$$W_B = W_S = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)v_0].$$

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

$$W_B = W_S = \gamma v_H + (1 - \gamma)v_0.$$

Case 1.4. $c < \gamma(v_L - v_0)$.

$$W_B = W_S = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c].$$

Part (ii).

In this case, we have $c \in (\gamma(v_H - v_L), \gamma(v_H - v_0))$.

When $c_B \in (0, \underline{C}) \cup (\overline{C}, +\infty)$, by Proposition 2, we know that both sellers will use the same pure strategy. Hence, $W_S = W_B$.

When $c_B \in (\underline{C}, \overline{C})$, we show W_B is decreasing in c_B .

Note that in this case, by the proof of Proposition 2, we have:

$$\begin{aligned} \beta_H &= \frac{c_B Q - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right)}{\frac{R\delta(p_2 - p_1)}{1 - \delta} \frac{(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}}, \\ \beta_L &= \frac{c_B Q - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right)}{\frac{R\delta(p_1 - p_2)}{1 - \delta} \frac{(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}}. \end{aligned}$$

We write $A \triangleq \frac{R\delta(p_1 - p_2)}{1 - \delta} \frac{(1 - p_1)(r_3 - \delta)(1 - r_2^Q) + p_2 r_3(1 - \delta r_2)(r_3^{-Q} - 1)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}$ for ease of notation. The expression of W_B is:

$$\begin{aligned} W_B &\triangleq [\beta_H \beta_L + (1 - \beta_H)(1 - \beta_L)] W_S \\ &\quad + \beta_H(1 - \beta_L)[P(Q)U_H + (1 - P(Q))U_L] + \beta_L(1 - \beta_H)[P(-Q)U_H + (1 - P(-Q))U_L]. \end{aligned}$$

Taking the derivative of W_B with respect to c_B gives:

$$\begin{aligned} \frac{dW_B(c_B)}{dc_B} &\propto - \left\{ \left[\frac{R(p_1 - p_2)}{1 - \delta} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)r_2^Q + p_2 r_3(1 - \delta r_2)r_3^{-Q}]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) - 2Qc_B \right] \right. \\ &\quad \cdot [P(Q) + P(-Q) - P(\bar{0}) - P(\underline{0})](U_H - U_L) + [P(Q) - P(-Q)](U_H - U_L)A \Big\} \\ &= -(U_H - U_L) \left\{ A[1 - P(-Q)] \right. \\ &\quad \left. + [P(-Q) - P(\underline{0})] \left[\frac{R(p_1 - p_2)}{1 - \delta} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)r_2^Q + p_2 r_3(1 - \delta r_2)r_3^{-Q}]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) - 2Qc_B \right] \right\} \end{aligned}$$

We write $A' \triangleq \frac{R(p_1 - p_2)}{1 - \delta} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)r_2^Q + p_2 r_3(1 - \delta r_2)r_3^{-Q}]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right)$ for ease of notation.

Hence, to show $\frac{dW_B(c_B)}{dc_B} \leq 0$, it is equivalent to showing that $A[1 - P(-Q)] + [P(-Q) - P(\underline{0})](A' - 2Qc_B) > 0$, which is equivalent to $P(-Q)[A' - A - 2Qc_B] + A - P(\underline{0})(A' - 2Qc_B) > 0$. Note that

$$A' \geq 2Qc_B \Leftrightarrow c_B \leq \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)r_2^Q + p_2r_3^{1-Q}(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \triangleq \tilde{C} \text{ defined in (A.17).}$$

• **When $\underline{C} \leq c_B \leq \tilde{C}$** , then we have

$$\begin{aligned} P(-Q)[A' - A - 2Qc_B] + A - P(\underline{0})(A' - 2Qc_B) &\geq P(-Q)[A' - A - 2Qc_B] + A - (A' - 2Qc_B) \\ &= (A' - A - 2c_B Q)[P(-Q) - 1]. \end{aligned}$$

Note that

$$A' - A - 2Qc_B = \frac{R(p_1 - p_2)}{1 - \delta} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) - 2c_B Q \leq 0 \Leftrightarrow c_B \geq \underline{C}.$$

Hence, $(A' - A - 2c_B Q)[P(-Q) - 1] \geq 0$, as $P(-Q) \leq 1$.

• **When $\tilde{C} < c_B \leq \overline{C}$** , then we have

$$\begin{aligned} P(-Q)[A' - A - 2Qc_B] + A - P(\underline{0})(A' - 2Qc_B) &= P(-Q)[A' - A - 2Qc_B] + A + P(\underline{0})(2Qc_B - A') \\ &\geq P(-Q)[A' - A - 2Qc_B] + A + P(-Q)(2Qc_B - A') = A[1 - P(-Q)] \geq 0. \end{aligned}$$

Hence, W_B is decreasing in c_B .

Now, we study the relationship between W_S and W_B , W_R and W_B at the end points $c_B \in \{\underline{C}, \overline{C}\}$.

When $c_B = \underline{C}$, we have $\beta_H = 1$ and $\beta_L = 0$ by Proposition 2. Hence, it is easy to verify that

$$W_B > W_S \Leftrightarrow P(Q) - \frac{P(\overline{0}) + P(\underline{0})}{2} > 0.$$

Note that by Lemma 1, we know that $P(s)$ is increasing in s . Thus,

$$P(Q) - \frac{P(\overline{0}) + P(\underline{0})}{2} \geq P(\overline{0}) \frac{P(\overline{0}) + P(\underline{0})}{2} = \frac{P(\overline{0}) - P(\underline{0})}{2} \geq 0,$$

and the inequality is strict if $p_2 < 1/2$.

Hence, by continuity of β_H and β_L as functions of c_B , we know that there exists $\underline{C}'_1 \in (\underline{C}, \overline{C})$ such that $W_S < W_B$ if and only if $c_B < \underline{C}'_1$.

When $c_B = \overline{C}$, we have $\beta_L = 1$ and $\beta_H = 0$ by Proposition 2. Hence, it is easy to verify that

$$W_B < W_R \Leftrightarrow P(-Q) < \frac{1}{2}, \quad (\text{A.19})$$

which is ensured by $P(-Q) = \frac{1-\gamma^2}{1+\gamma^2} \left(\frac{1-p_2}{p_2} \right)^{-Q}$, where $p_2 = \frac{1-\gamma^2}{2}$ and $\gamma > \sqrt{\frac{1-2-Q-1}{1+2-Q-1}}$. We define $\bar{\gamma} \triangleq \sqrt{\frac{1-2-Q-1}{1+2-Q-1}}$. Hence, by continuity of β_H and β_L as functions of c_B , we know that there exists $\overline{C}'_1 \in (\underline{C}, \overline{C})$ such that $W_R > W_B$ whenever $C \in (\overline{C}'_1, \overline{C})$ and $\gamma > \sqrt{\frac{1-2-Q-1}{1+2-Q-1}}$. When $\gamma \leq \sqrt{\frac{1-2-Q-1}{1+2-Q-1}}$, then $\overline{C}'_1 \triangleq \overline{C}$.

Because W_B is decreasing in $c_B \in (\underline{C}, \overline{C})$ and $W_S > W_R$ (both being independent of c_B), it follows that

- (a) $W_B > W_S > W_R$ if and only if $\underline{C} < c_B < \underline{C}'_1$;
- (b) $W_S > W_B > W_R$ if and only if $\underline{C}'_1 < c_B < \overline{C}'_1$.
- (c) $W_S > W_R > W_B$ if and only if $\overline{C}'_1 < c_B < \overline{C}$. $\square$

Proof of Theorem 3

Part (i).

We explicitly write customer welfare under the three ranking systems as a function of the search cost.

Based on the proof of Theorem 1, we know that:

Case 1. $v_H + v_0 \geq 2v_L$. In this case, we have $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Let $c_1 \geq \gamma(v_H - v_0)$, $\gamma(v_H - v_L) \leq c_2 < \gamma(v_H - v_0)$, $\gamma(v_L - v_0) \leq c_3 < \gamma(v_H - v_L)$ and $c_4 < \gamma(v_L - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$.

$$W_R = W_S = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)v_0].$$

Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$.

$$W_R = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]],$$

$$W_S = \frac{1}{2} \left[\gamma v_H + (1 - \gamma)v_0 + \frac{1 - \gamma^2}{1 + \gamma^2}[\gamma v_H + (1 - \gamma)v_0] + \frac{2\gamma^2}{1 + \gamma^2}\{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\} \right].$$

We have:

$$W_R(c_2) > W_R(c_1) \Leftrightarrow \gamma v_H + (1 - \gamma)v_0 - c > v_0 \Leftrightarrow c < \gamma(v_H - v_0),$$

$$W_S(c_2) > W_S(c_1) \Leftrightarrow \frac{1 - \gamma^2}{1 + \gamma^2}[\gamma v_H + (1 - \gamma)v_0] + \frac{2\gamma^2}{1 + \gamma^2}\{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\} > \gamma v_L + (1 - \gamma)v_0$$

$$\Leftrightarrow \frac{1 - \gamma^2}{1 + \gamma^2}(v_H - \gamma v_L) + \frac{2\gamma^2}{1 + \gamma^2}(1 - \gamma)[\gamma(v_H - v_0) - c] > 0,$$

which is true when $c < \gamma(v_H - v_0)$.

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

$$W_R = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]],$$

$$W_S = \gamma v_H + (1 - \gamma)v_0.$$

We have:

$$W_R(c_3) > W_R(c_2) \Leftrightarrow \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0] - c > \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]$$

$$\Leftrightarrow \gamma^2(v_H - v_L) - \gamma c > 0,$$

which is true when $c < \gamma(v_H - v_L)$.

We also have

$$W_S(c_3) > W_S(c_2) \Leftrightarrow \gamma v_H + (1 - \gamma)v_0 > \frac{1 - \gamma^2}{1 + \gamma^2}[\gamma v_H + (1 - \gamma)v_0] + \frac{2\gamma^2}{1 + \gamma^2}\{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\}$$

$$\Leftrightarrow \gamma v_H + (1 - \gamma)v_0 - \{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\} > 0 \Leftrightarrow \gamma(v_H - v_L) + (1 - \gamma)(-\gamma v_H + \gamma v_0 + c) > 0.$$

Note that as $c \geq \gamma(v_L - v_0)$, hence we have

$$\gamma(v_H - v_L) + (1 - \gamma)(-\gamma v_H + \gamma v_0 + c) > \gamma(v_H - v_L) + (1 - \gamma)\gamma(v_L - v_0 - v_H + v_0) > 0.$$

Case 1.4. $c < \gamma(v_L - v_0)$.

$$W_R = \frac{1}{2}[\gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c] + \gamma[\gamma v_H + (1 - \gamma)v_L] + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]],$$

$$W_S = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c].$$

We have:

$$\begin{cases} W_S(c_4) > W_S(c_3), \\ W_S(c_4) > W_S(c_3), \end{cases} \Leftrightarrow \gamma v_L + (1-\gamma)v_0 - c > v_0, \Leftrightarrow c < \gamma(v_L - v_0).$$

Part (ii).

Based on the proof of Theorem 2 we know that

Case 1. $v_H + v_0 \geq 2v_L$. In this case, we have $\gamma(v_L - v_0) \leq \gamma(v_H - v_L) < \gamma(v_H - v_0)$.

Case 1.1. $c \geq \gamma(v_H - v_0)$.

$$W_B = \frac{1}{2}[\gamma v_H + (1-\gamma)v_0 + \gamma v_L + (1-\gamma)v_0].$$

Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$.

$$\begin{aligned} W_B = & \frac{1}{2}[(1-\beta_L)(1-\beta_H) + \beta_H\beta_L][P(\bar{0})U_H + (1-P(\bar{0}))U_L + P(\underline{0})U_H + (1-P(\underline{0}))U_L] \\ & + \beta_H(1-\beta_L)[P(Q)U_H + (1-P(Q))U_L] + \beta_L(1-\beta_H)[P(-Q)U_H + (1-P(-Q))U_L], \end{aligned}$$

where

$$\begin{cases} U_H = \gamma v_H + (1-\gamma)v_0, \\ U_L = \gamma v_L + (1-\gamma)[\gamma v_H + (1-\gamma)v_0 - c], \end{cases}, \quad P(\bar{0}) = P(Q) = 1, \quad P(\underline{0}) = \frac{1-\gamma^2}{1+\gamma^2}, \quad P(-Q) = \left(\frac{1-\gamma^2}{1+\gamma^2}\right)^{Q+1},$$

as $P(-Q) = \left(\frac{1-\gamma^2}{1+\gamma^2}\right) \left(\frac{1-p_2}{p_2}\right)^{-Q}$ where $p_2 = \frac{1-\gamma^2}{2}$.

Case 1.3. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$.

$$W_B = \gamma v_H + (1-\gamma)v_0.$$

Case 1.4. $c < \gamma(v_L - v_0)$.

$$W_B = \gamma v_H + (1-\gamma)[\gamma v_L + (1-\gamma)v_0 - c].$$

It is clear that when $c < \gamma(v_H - v_L)$, W_B decreases in c . Let $c' \geq \gamma(v_H - v_0)$ and $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$. Now we show that there exists $\bar{C}'_1 \in (\underline{C}, \bar{C})$ and $\bar{\gamma}' \in (0, 1)$ such that $\forall c_B \in (\bar{C}'_1, \bar{C})$ and $\forall \gamma \in (\bar{\gamma}', 1)$, we have $W(c') > W(c)$, where $\underline{C}$ and $\bar{C}$ are defined in Theorem 2.

When $c_B = \bar{C}$, then $\beta_H = 0, \beta_L = 1$, so we have:

$$\begin{aligned} W_B(c') &= \frac{1}{2}[\gamma v_H + (1-\gamma)v_0] + \frac{1}{2}[\gamma v_L + (1-\gamma)v_0], \\ W_B(c) &= P(-Q)[\gamma v_H + (1-\gamma)v_0] + [1 - P(-Q)]\{\gamma v_L + (1-\gamma)[\gamma v_H + (1-\gamma)v_0 - c]\}. \end{aligned}$$

Then we have

$$\begin{aligned} W_B(c') - W_B(c) &= [\gamma v_H + (1-\gamma)v_0][1/2 - P(-Q)] + \gamma v_L[P(-Q) - 1/2] \\ &\quad + (1-\gamma)\{v_0/2 - [1 - P(-Q)][\gamma v_H + (1-\gamma)v_0 - c]\} \\ &= [1/2 - P(-Q)][\gamma v_H + (1-\gamma)v_0 - \gamma v_L] + (1-\gamma)\{v_0/2 - [1 - P(-Q)][\gamma v_H + (1-\gamma)v_0 - c]\}. \end{aligned}$$

It is clear that when $\gamma = 1$, the above expression is strictly positive. Hence, by continuity, there exists $\bar{\gamma}_1 \in (0, 1)$ and $\bar{C}'_1 \in (\underline{C}, \bar{C})$ such that $\forall c_B \in (\bar{C}'_1, \bar{C})$ and $\forall \gamma \in (\bar{\gamma}_1, 1)$, we have $W_B(c') > W_B(c)$. $\square$

Proof of Proposition 3

Due to the nature of the proof, we will show (i), (ii) and (iii) together. The explicit expression of C_0 is given in (A.23), from which, it is easy to see that $C_0 = 0$ if and only if $p_1 = p_2$.

We will show $(Q_H, Q_L) = (0, 0)$ is the unique pure strategy equilibrium if and only if $c_B \geq C_0$ in the last part of this proof.

Based on the value function $V_i(s)$ in (A.8), the expected discounted profit for seller $i \in \{H, L\}$, given seller H brushing Q_H and seller L brushing Q_L is denoted by $\pi_i(Q_H; Q_L)$:

$$\begin{aligned} & \pi_H(Q_H; Q_L) \\ = & \begin{cases} \frac{Rp_1}{1-\delta} + \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_2^{Q_H-Q_L} - c_B Q_H, & \text{if } Q_H > Q_L \geq 0, \\ \frac{1}{2} \left(\frac{R(p_1+p_2)}{1-\delta} + \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right) - c_B Q_H, & \text{if } Q_H = Q_L \geq 0, \\ \frac{Rp_2}{1-\delta} - \frac{R\delta p_2 r_3(p_1-p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_3^{Q_H-Q_L} - c_B Q_H, & \text{if } 0 \leq Q_H < Q_L. \end{cases} \\ & \pi_L(Q_L; Q_H) \\ = & \begin{cases} \frac{R(1-p_1)}{1-\delta} - \frac{R\delta(p_1-p_2)(1-p_1)(r_3-\delta)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_2^{Q_H-Q_L} - c_B Q_L, & \text{if } Q_H > Q_L \geq 0, \\ \frac{1}{2} \left(\frac{R(2-p_1-p_2)}{1-\delta} - \frac{R\delta(p_1-p_2)[(1-p_1)(r_3-\delta) - p_2 r_3(1-\delta r_2)]}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} \right) - c_B Q_L, & \text{if } Q_H = Q_L \geq 0, \\ \frac{R(1-p_2)}{1-\delta} + \frac{R\delta p_2 r_3(p_1-p_2)(1-\delta r_2)}{(1-\delta)[(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)]} r_3^{Q_H-Q_L} - c_B Q_L, & \text{if } 0 \leq Q_H < Q_L. \end{cases} \end{aligned}$$

Part (i)

Case 1. Assuming $Q_H > Q_L \geq 0$ is an equilibrium.

In this case, it implies for seller H :

$$\begin{aligned} & \pi_H(Q_H; Q_L) \geq \pi_H(Q_L; Q_L) \Leftrightarrow \\ & c_B(Q_H - Q_L) < \frac{R(p_1 - p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^{Q_H-Q_L} - 1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right). \end{aligned}$$

And for seller L :

$$\begin{aligned} & \pi_L(Q_L; Q_H) \geq \pi_L(Q_H; Q_H) \Leftrightarrow \\ & c_B(Q_H - Q_L) > \frac{R(p_1 - p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^{Q_H-Q_L} - 1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right). \end{aligned}$$

Therefore, the necessary condition for $Q_H > Q_L \geq 0$ being an equilibrium is that:

$$c_B = \frac{R(p_1 - p_2)}{2(1-\delta)(Q_H - Q_L)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^{Q_H-Q_L} - 1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right). \quad (\text{A.20})$$

Case 1.1. $Q_H - Q_L > 1$.

In this case, we show $\pi_H(Q_L + 1; Q_L) > \pi_H(Q_L; Q_L)$ and the last quantity equals $\pi_H(Q_H; Q_L)$ by (A.20).

We have:

$$\begin{aligned} \pi_H(Q_L + 1; Q_L) - \pi_H(Q_L; Q_L) &= \frac{R(p_1 - p_2)}{2(1-\delta)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2 - 1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1-\delta)(Q_H - Q_L)} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^{Q_H-Q_L} - 1) + p_2 r_3(1-\delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2-1) + r_3) + \delta^2 p_2 r_2(1-p_1)} \right). \end{aligned}$$

Note that obviously, when $Q_H - Q_L = 1$, then $\pi_H(Q_L + 1; Q_L) - \pi_H(Q_L; Q_L) = 0$. Now we define:

$$F(D) \triangleq \frac{1}{D} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^D-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right).$$

To show $\pi_H(Q_L + 1; Q_L) - \pi_H(Q_L; Q_L) > 0$ for $Q_H - Q_L > 1$, it suffices to show that $F(D)$ is decreasing in D .

Note that

$$\frac{dF(D)}{dD} \propto -\frac{1}{D} \left(1 + \frac{\delta[(1-p_1)(r_3-\delta)(2r_2^D-1) + p_2r_3(1-\delta r_2)]}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)} \right) + \frac{\delta(1-p_1)(r_3-\delta)2r_2^D \ln r_2}{(\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1)}.$$

Hence,

$$\begin{aligned} \frac{dF(D)}{dD} < 0 &\Leftrightarrow D\delta(1-p_1)(r_3-\delta)2r_2^D \ln r_2 \\ &> (\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1) + \delta[(1-p_1)(r_3-\delta)(2r_2^D-1) + p_2r_3(1-\delta r_2)] \\ &\Leftrightarrow \delta(1-p_1)(r_3-\delta)2r_2^D(D \ln r_2 - 1) \\ &> (\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1) + \delta[p_2r_3(1-\delta r_2) - (1-p_1)(r_3-\delta)]. \end{aligned}$$

It is easy to verify that $r_2^D(D \ln r_2 - 1)$ is increasing in D (taking the first order derivative with respect to D). Hence, it suffices to show that the above inequality is true when $D = 1$.

$$\begin{aligned} \delta(1-p_1)(r_3-\delta)2r_2(\ln r_2 - 1) &> (\delta p_1r_2-1)(\delta(p_2-1)+r_3) + \delta^2p_2r_2(1-p_1) + \delta[p_2r_3(1-\delta r_2) - (1-p_1)(r_3-\delta)] \\ &\Leftrightarrow \delta(1-p_1)(r_3-\delta)2r_2(\ln r_2 - 1) > (\delta - r_3)(1 - \delta p_1r_2) - \delta p_2(1 - \delta r_2) + \delta[p_2r_3(1 - \delta r_2) - (1 - p_1)(r_3 - \delta)] \\ &\Leftrightarrow \delta(1-p_1)(r_3-\delta)2r_2(\ln r_2 - 1) > (\delta - r_3)(1 - \delta p_1r_2 + \delta - \delta p_1) - \delta p_2(1 - \delta r_2)(1 - r_3) \\ &\Leftrightarrow (r_3 - \delta)[\delta(1-p_1)(2r_2(\ln r_2 - 1) + 1) + 1 - \delta p_1r_2] > \delta p_2(1 - \delta r_2)(r_3 - 1). \end{aligned}$$

Note that $r_3 - \delta > r_3 - 1$. Thus, it suffices to show that

$$\delta(1-p_1)[2r_2(\ln r_2 - 1) + 1] + 1 - \delta p_1r_2 - \delta p_2(1 - \delta r_2) > 0.$$

Note that $2r_2(\ln r_2 - 1) + 1 > -1$. Hence

$$\begin{aligned} \delta(1-p_1)[2r_2(\ln r_2 - 1) + 1] + 1 - \delta p_1r_2 - \delta p_2(1 - \delta r_2) &> -\delta(1-p_1) + 1 - \delta p_1r_2 - \delta p_2(1 - \delta r_2) \\ &= 1 - \delta + \delta(p_1 - p_2) - \delta r_2(p_1 - \delta p_2) = 1 - \delta + \delta[p_1(1 - r_2) - p_2(1 - r_2\delta)] > 1 - \delta + \delta[p_2(1 - r_2) - p_2(1 - r_2\delta)] \\ &= 1 - \delta + p_2r_2(\delta - 1) = (1 - \delta)(1 - p_2r_2) > 0. \end{aligned}$$

Therefore this contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Case 1.2. $Q_H - Q_L = 1$.

In this case, under (A.20), we have $\pi_L(Q_L; Q_H) = \pi_L(Q_H; Q_H)$. Thus,

$$\begin{aligned} \pi_L(Q_H + 1; Q_H) - \pi_L(Q_H; Q_H) &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2p_2r_2(1 - p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2r_3(1 - \delta r_2)]}{(\delta p_1r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2p_2r_2(1 - p_1)} \right) \\ &= \frac{R(p_1 - p_2)\delta}{1 - \delta} \frac{p_2r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2)}{(\delta p_1r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2p_2r_2(1 - p_1)} \end{aligned}$$

Note that because of (A.9), we have:

$$\begin{aligned} \pi_L(Q_H + 1; Q_H) - \pi_L(Q_H; Q_H) &> 0 \Leftrightarrow p_2 r_3 (1 - \delta r_2) (r_3^{-1} - 1) + (1 - p_1) (r_3 - \delta) (1 - r_2) < 0 \\ \Leftrightarrow \frac{(1 - p_1) (1 - r_2)}{1 - \delta r_2} &< \frac{p_2 (1 - r_3^{-1})}{1 - \delta / r_3} \Leftrightarrow p_1 + p_2 > 1, \text{ by (A.11)}. \end{aligned}$$

Therefore, this contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Case 2. Assuming $Q_L > Q_H \geq 0$ is an equilibrium.

In this case, for seller L ,

$$\begin{aligned} \pi_L(Q_L; Q_H) &\geq \pi_L(Q_H; Q_H) \Leftrightarrow \\ c_B(Q_L - Q_H) &\leq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{Q_H - Q_L} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right). \end{aligned}$$

However, for seller H ,

$$\begin{aligned} \pi_H(Q_H; Q_L) &\geq \pi_H(Q_L; Q_L) \Leftrightarrow \\ c_B(Q_L - Q_H) &\geq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{Q_H - Q_L} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right). \end{aligned}$$

Therefore, the necessary condition for $Q_L > Q_H \geq 0$ being an equilibrium is that:

$$c_B = \frac{R(p_1 - p_2)}{2(1 - \delta)(Q_L - Q_H)} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{Q_H - Q_L} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right). \quad (\text{A.21})$$

Case 2.1. $Q_L - Q_H > 1$.

In this case, we show $\pi_L(Q_H + 1; Q_H) > \pi_L(Q_H; Q_H)$ and the last quantity equals $\pi_L(Q_L; Q_H)$ by (A.21).

We have:

$$\begin{aligned} \pi_L(Q_H + 1; Q_H) - \pi_L(Q_H; Q_H) &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{-1} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)(Q_L - Q_H)} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{Q_H - Q_L} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right). \end{aligned}$$

Note that obviously, when $Q_L - Q_H = 1$, then $\pi_L(Q_H + 1; Q_H) - \pi_L(Q_H; Q_H) = 0$. Now we define:

$$G(D) \triangleq \frac{1}{D} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{-D} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right).$$

To show $\pi_L(Q_H + 1; Q_H) - \pi_L(Q_H; Q_H) > 0$ for $Q_L - Q_H > 1$, it suffices to show that $G(D)$ is decreasing in D . Note that

$$\frac{dG(D)}{dD} \propto -\frac{1}{D} \left(1 + \frac{\delta [p_2 r_3 (1 - \delta r_2) (2r_3^{-D} - 1) + (1 - p_1) (r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)} \right) + \frac{\delta p_2 r_3 (1 - \delta r_2) 2r_3^{-D} \ln(1/r_3)}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1)}.$$

Hence,

$$\begin{aligned} \frac{dG(D)}{dD} < 0 &\Leftrightarrow D \delta p_2 r_3 (1 - \delta r_2) 2r_3^{-D} \ln(1/r_3) \\ &> (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1) + \delta [p_2 r_3 (1 - \delta r_2) (2r_3^{-D} - 1) + (1 - p_1) (r_3 - \delta)] \\ &\Leftrightarrow \delta p_2 r_3 (1 - \delta r_2) 2r_3^{-D} (D \ln(1/r_3) - 1) \\ &> (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2 (1 - p_1) + \delta [(1 - p_1) (r_3 - \delta) - p_2 r_3 (1 - \delta r_2)]. \end{aligned}$$

It is easy to verify that $r_3^{-D}(D \ln(1/r_3) - 1)$ is increasing in D (taking the first order derivative with respect to D). Hence, it suffices to show that the above inequality is true when $D = 1$.

$$\begin{aligned}
& \delta p_2(1 - \delta r_2)2(\ln(1/r_3) - 1) > (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)] \\
& \Leftrightarrow \delta p_2(1 - \delta r_2)2(\ln(1/r_3) - 1) > (\delta - r_3)(1 - \delta p_1 r_2) - \delta p_2(1 - \delta r_2) + \delta[(1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)] \\
& \Leftrightarrow \delta p_2(1 - \delta r_2)2(\ln(1/r_3) - 1) > (\delta - r_3)(1 - \delta p_1 r_2 - \delta + \delta p_1) - \delta p_2(1 - \delta r_2)(1 + r_3) \\
& \Leftrightarrow \delta p_2(1 - \delta r_2)[2 \ln(1/r_3) + r_3 - 1] - (\delta - r_3)(1 - \delta p_1 r_2 - \delta + \delta p_1) > 0.
\end{aligned}$$

Note that when $r_3 = 1$,

$$\delta p_2(1 - \delta r_2)[2 \ln(1/r_3) + r_3 - 1] - (\delta - r_3)(1 - \delta p_1 r_2 - \delta + \delta p_1) = (1 - \delta)[1 - \delta + \delta p_1(1 - r_2)] > 0$$

Hence, we define $Y(r_3) \triangleq \delta p_2(1 - \delta r_2)[2 \ln(1/r_3) + r_3 - 1] - (\delta - r_3)(1 - \delta p_1 r_2 - \delta + \delta p_1)$. Note that

$$\frac{dY(r_3)}{dr_3} = -2\delta p_2(1 - \delta r_2)\frac{1}{r_3} + \delta p_2(1 - \delta r_2) + [1 - \delta + \delta p_1(1 - r_2)]$$

Hence, $\frac{dY(r_3)}{dr_3}$ is increasing in r_3 . Thus,

$$\begin{aligned}
\left. \frac{dY(r_3)}{dr_3} \right|_{r_3=1} &= -2\delta p_2(1 - \delta r_2) + \delta p_2(1 - \delta r_2) + [1 - \delta + \delta p_1(1 - r_2)] \\
&= -\delta p_2(1 - \delta r_2) + [1 - \delta + \delta p_1(1 - r_2)] \\
&= 1 - \delta + \delta p_1(1 - r_2) - \delta p_2(1 - \delta r_2) \\
&> 1 - \delta + \delta p_1(1 - r_2) - \delta p_1(1 - \delta r_2) \\
&= (1 - \delta)(1 - r_2 \delta p_1) > 0.
\end{aligned}$$

Therefore, this contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Case 2.2. $Q_L - Q_H = 1$.

In this case, under (A.21), we have $\pi_H(Q_H; Q_L) = \pi_H(Q_L; Q_L)$.

Case 2.2.1. $Q_L > 1$.

$$\begin{aligned}
\pi_H(0; Q_L) - \pi_H(Q_L; Q_L) &= \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)(2r_3^{-Q_L} - 1)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\
&\quad + \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_L \\
&\triangleq Z(Q_L)
\end{aligned}$$

It is clear that when $Q_L = 1$, $\pi_H(0; Q_L) - \pi_H(Q_L; Q_L) = 0$. To show $Z(Q) > 0$ for $Q > 1$, it suffices to show that $Z(Q)$ is increasing in Q . Note that

$$\frac{dZ(Q)}{dQ} = \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) - p_2 r_3(1 - \delta r_2)2r_3^{-Q} \ln(1/r_3)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right).$$

Hence,

$$\begin{aligned}
\frac{dZ(Q)}{dQ} > 0 &\Leftrightarrow (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) \\
&\quad + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) + p_2 r_3(1 - \delta r_2)2r_3^{-Q} \ln r_3] > 0.
\end{aligned}$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)] > 0,$$

which is true by (A.13). This contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Case 2.2.2. $Q_L = 1$.

In this case, we show $(Q_H, Q_L) = (0, 1)$ is an equilibrium.

Under (A.21), we know that $\pi_H(1; 1) = \pi_H(0; 1)$. Now we show $\pi_H(Q_H; 1)$ is decreasing in Q_H for $Q_H > 1$.

We have:

$$\begin{aligned} \pi_H(Q_H; 1) &= \frac{R p_1}{1 - \delta} + \frac{R \delta (p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H - 1}}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_H. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d\pi_H(Q_H; 1)}{dQ_H} &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R \delta (p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H - 1} \ln r_2}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - 2r_2^{Q_H - 1} \ln r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d\pi_H(Q_H; 1)}{dQ_H} < 0 &\Leftrightarrow (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) \\ &\quad + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - 2r_2^{Q_H - 1} \ln r_2)] > 0. \end{aligned}$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)] > 0,$$

which is true by (A.13).

Now we show $\pi_H(2; 1) < \pi_H(1; 1)$. We have

$$\pi_H(1; 1) - \pi_H(2; 1) = \frac{R(p_1 - p_2)}{1 - \delta} \frac{\delta[p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}.$$

Hence,

$$\begin{aligned} \pi_H(1; 1) - \pi_H(2; 1) > 0 &\Leftrightarrow p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2) < 0 \\ &\Leftrightarrow \frac{(1 - p_1)(1 - r_2)}{1 - \delta r_2} < \frac{p_2(1 - r_3^{-1})}{1 - \delta/r_3} \Leftrightarrow p_1 + p_2 > 1, \text{ by (A.11)}. \end{aligned}$$

Therefore, given $Q_L = 1$, the best response of seller H is $Q_H \in \{0, 1\}$.

Now we show when $Q_H = 0$, then $Q_L = 1$ is the best response of seller L . Under (A.21), we know that $\pi_L(1; 0) = \pi_L(0; 0)$. Now we show that $\pi_L(Q_L; 0)$ is decreasing in Q_L for $Q_L \geq 1$. We have

$$\begin{aligned} \pi_L(Q_L; 0) &= \frac{R(1 - p_2)}{1 - \delta} + \frac{R \delta p_2 r_3 (p_1 - p_2)(1 - \delta r_2) r_3^{-Q_L}}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_L. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d\pi_L(Q_L; 0)}{dQ_L} &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) - 2p_2 r_3(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d\pi_L(Q_L; 0)}{dQ_L} < 0 &\Leftrightarrow (1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)] \\ &\quad + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) - 2p_2 r_3(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)] > 0. \end{aligned}$$

It suffices to show that

$$(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)] + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)] > 0,$$

which is true by (A.13). Therefore, given $Q_H = 0$, the best response of seller L is $Q_L \in \{0, 1\}$.

Case 3. Assuming $Q_H = Q_L = Q > 0$ is an equilibrium.

In this case, for seller H :

$$\pi_H(Q; Q) \geq \pi_H(Q + 1; Q) \Leftrightarrow c_B \geq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right).$$

For seller L :

$$\pi_L(Q; Q) \geq \pi_L(Q - 1; Q) \Leftrightarrow c_B \leq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right).$$

Therefore, the necessary condition for $Q_H = Q_L = Q > 0$ being an equilibrium is that:

$$c_B = \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \quad (\text{A.22})$$

Note that

$$\begin{aligned} \pi_L(Q + 1; Q) - \pi_L(Q; Q) &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= \frac{R(p_1 - p_2)\delta[p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]}. \end{aligned}$$

Hence,

$$\begin{aligned} \pi_L(Q + 1; Q) - \pi_L(Q; Q) > 0 &\Leftrightarrow p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2) < 0 \\ &\Leftrightarrow \frac{(1 - p_1)(1 - r_2)}{1 - \delta r_2} < \frac{(1 - r_3^{-1})p_2}{1 - \delta/r_3} \Leftrightarrow p_1 + p_2 > 1, \end{aligned}$$

which is proved in (A.11). Therefore, this contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Part (ii)

We have $p_1 + p_2 = 1$. Based on the analysis in (i), we know:

Case 1.2. $Q_H = Q_L = 1$.

In this case, under (A.20), we have $\pi_L(Q_L; Q_H) = \pi_L(Q_H; Q_H)$.

Case 1.2.1. $Q_H > 1$.

$$\begin{aligned} \pi_L(0; Q_H) - \pi_L(Q_H; Q_H) &= \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^{Q_H} - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_H \\ &\triangleq T(Q_H). \end{aligned}$$

It is clear that when $Q_H = 1$, then we have $\pi_L(0; Q_H) - \pi_L(Q_H; Q_H) = 0$. To show that $T(Q) > 0$ for $Q > 1$, it suffices to show that $T(Q)$ is increasing in Q .

Note that

$$\frac{dT(Q)}{dQ} = \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - (1 - p_1)(r_3 - \delta)2r_2^{Q_H} \ln r_2]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right).$$

Hence,

$$\begin{aligned} \frac{dT(Q)}{dQ} > 0 &\Leftrightarrow (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) \\ &\quad + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - (1 - p_1)(r_3 - \delta)2r_2^{Q_H} \ln r_2] > 0 \end{aligned}$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)] > 0,$$

which is true by (A.14).

Therefore, this contradicts with the assumption that (Q_H, Q_L) is an equilibrium.

Case 1.2.2. $Q_H = 1$.

In this case, we show $(Q_H, Q_L) = (1, 0)$ is an equilibrium.

Under (A.20), we have $\pi_L(0; 1) = \pi_L(1; 1)$. Now we show that $\pi_L(Q_L; 1)$ is decreasing in Q_L for $Q_L > 1$.

We have:

$$\begin{aligned} \pi_L(Q_L; 1) &= \frac{R(1 - p_2)}{1 - \delta} + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{1-Q_L}}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_L. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d\pi_L(Q_L; 1)}{dQ_L} &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{1-Q_L} \ln(1/r_3)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - p_2 r_3(1 - \delta r_2)r_3^{1-Q_L} \ln r_3]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d\pi_L(Q_L; 1)}{dQ_L} < 0 &\Leftrightarrow (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) \\ &\quad + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - p_2 r_3(1 - \delta r_2)r_3^{1-Q_L} \ln(1/r_3)] > 0. \end{aligned}$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)] > 0.$$

which is true by (A.14).

Now we show $\pi_L(2; 1) = \pi_L(1; 1)$. We have

$$\pi_L(1; 1) - \pi_L(2; 1) = \frac{R(p_1 - p_2)}{1 - \delta} \frac{\delta[(1 - p_1)(r_3 - \delta)(r_2 - 1) + p_2 r_3(1 - \delta r_2)(1 - r_3^{-1})]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}.$$

Note that

$$(1 - p_1)(r_3 - \delta)(r_2 - 1) + p_2 r_3(1 - \delta r_2)(1 - r_3^{-1}) = 0 \Leftrightarrow p_1 + p_2 = 1, \text{ by (A.11).}$$

Hence, given $Q_H = 1$, seller L 's best response is $\{0, 1, 2\}$.

Now we show that when $Q_L = 0$, then $Q_H = 1$ is the best response of seller H . Under (A.20), we know that $\pi_H(1; 0) = \pi_H(0; 0)$. Now we show that $\pi_H(Q_H; 0)$ is decreasing in Q_H for $Q_H \geq 1$. We have:

$$\begin{aligned} \pi_H(Q_H; 0) &= \frac{R p_1}{1 - \delta} + \frac{R \delta (p_1 - p_2)(1 - p_1)(r_3 - \delta)}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} r_2^{Q_H} \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) Q_H. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d\pi_H(Q_H; 0)}{dQ_H} &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R \delta (p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H} \ln r_2}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - (1 - p_1)(r_3 - \delta) r_2^{Q_H} \ln r_2]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d\pi_H(Q_H; 0)}{dQ_H} < 0 &\Leftrightarrow (\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) \\ &\quad + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2) - (1 - p_1)(r_3 - \delta) r_2^{Q_H} \ln r_2] > 0. \end{aligned}$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)] > 0,$$

which is true by (A.14). Therefore, given $Q_L = 0$, the best response for seller H is $Q_H \in \{0, 1\}$.

Case 3. $Q_H = Q_L = Q > 0$.

Case 3.1. $Q_H = Q_L = Q > 1$.

The proof follows the same logic as Case 1.2.1.

Case 3.2. $Q_H = Q_L = Q = 1$.

In this case, we show $(Q_H, Q_L) = (1, 1)$ is an equilibrium. We first show that under (A.22), $\pi_L(Q_L; 1)$ is decreasing in Q_L for $Q_L > 1$ and then we show that $\pi_L(2; 1) = \pi_L(1; 1)$. The proof is exactly the same as Case 1.2.2.

Then, we show that given $Q_L = 1$, $\pi_H(Q_H; 1)$ is decreasing in Q_H for $Q_H \geq 2$. Note that under (A.22), we have $\pi_H(2; 1) = \pi_H(1; 1)$. Hence, we only need to show that $\pi_H(0; 1) \leq \pi_H(1; 1)$. We have

$$\begin{aligned} \pi_H(1; 1) - \pi_H(0; 1) &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) = 0. \end{aligned}$$

Hence $(Q_H, Q_L) = (1, 1)$ is an equilibrium.

Now we show that $(Q_H, Q_L) = (0, 0)$ is an equilibrium if and only if

$$c_B \geq \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \triangleq C_0. \quad (\text{A.23})$$

It is clear that $C_0 = 0$ if and only if $p_1 = p_2$.

We have already shown that when (A.23) holds with equality, then $(Q_H, Q_L) = (0, 0)$ is an equilibrium.

If (A.23) holds strictly, given $Q_L = 0$, then $\pi_H(Q_H; 0)$ is decreasing in Q_H for $Q_H \geq 1$. We have:

$$\pi_H(Q_H; 0) = \frac{R p_1}{1 - \delta} + \frac{R \delta(p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H - 1}}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} - c_B Q_H.$$

Note that

$$\begin{aligned} \frac{d\pi_H(Q_H; 1)}{dQ_H} &= -c_B + \frac{R \delta(p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H - 1} \ln r_2}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &< -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R \delta(p_1 - p_2)(1 - p_1)(r_3 - \delta) r_2^{Q_H - 1} \ln r_2}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \\ &= -\frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - 2r_2^{Q_H - 1} \ln r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right). \end{aligned}$$

Hence, to show $\frac{d\pi_H(Q_H; 1)}{dQ_H} < 0$, it suffices to show that:

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - 2r_2^{Q_H - 1} \ln r_2)] > 0.$$

It suffices to show that

$$(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1) + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)] > 0,$$

which is true by (A.13).

Now we show $\pi_H(0; 0) > \pi_H(1; 0)$. We have:

$$\begin{aligned} \pi_H(0; 0) - \pi_H(1; 0) &= \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) + c_B \\ &> \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2 - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= \frac{R(p_1 - p_2)}{1 - \delta} \frac{\delta[p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)}. \end{aligned}$$

Note that

$$\begin{aligned} & \frac{R(p_1 - p_2)}{1 - \delta} \frac{\delta[p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \geq 0 \\ & \Leftrightarrow p_2 r_3(1 - \delta r_2)(r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)(1 - r_2) \leq 0 \\ & \Leftrightarrow p_1 + p_2 \geq 1. \end{aligned}$$

Hence, we know $\pi_H(0;0) - \pi_H(1;0) > 0$.

Now we show that given $Q_H = 0$, $\pi_L(Q_L;0)$ is decreasing in Q_L for $Q_L \geq 1$. We have:

$$\pi_L(Q_L;0) = \frac{R(1 - p_2)}{1 - \delta} + \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{-Q_L}}{(1 - \delta)(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} - c_B Q_L.$$

Note that

$$\begin{aligned} & \frac{d\pi_L(Q_L;0)}{dQ_L} \\ &= \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)}{(1 - \delta)(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} - c_B \\ &< \frac{R\delta p_2 r_3(p_1 - p_2)(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)}{(1 - \delta)(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) - 2p_2 r_3(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)]}{(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)]} \right). \end{aligned}$$

Hence, to show $\frac{d\pi_L(Q_L;0)}{dQ_L} < 0$, it suffices to show that:

$$\begin{aligned} & (1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)] \\ & + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta) - 2p_2 r_3(1 - \delta r_2)r_3^{-Q_L} \ln(1/r_3)] > 0. \end{aligned}$$

It suffices to show that

$$(1 - \delta)[(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)] + \delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)] > 0,$$

which is true by (A.13).

Now we show $\pi_L(0;0) > \pi_L(1;0)$. We have

$$\begin{aligned} \pi_L(0;0) - \pi_L(1;0) &= \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) + c_B \\ &> \frac{R(p_2 - p_1)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad + \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= 0. \end{aligned}$$

Hence, we know that $\pi_L(0;0) - \pi_L(1;0) > 0$.

Now we show that when (A.23) does not hold, then $(Q_H, Q_L) = (0, 0)$ is not an equilibrium.

Given $Q_H = 0$, then we have

$$\begin{aligned}\pi_L(1;0) - \pi_L(0;0) &= \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) - c_B \\ &> \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &\quad - \frac{R(p_1 - p_2)}{2(1 - \delta)} \left(1 + \frac{\delta[p_2 r_3(1 - \delta r_2)(2r_3^{-1} - 1) + (1 - p_1)(r_3 - \delta)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= 0.\end{aligned}$$

This implies $(Q_H, Q_L) = (0, 0)$ cannot be an equilibrium when (A.23) does not hold. $\square$

Appendix B: Proofs for Case $v_H + v_0 < 2v_L$

PROPOSITION B.1. *Customers' purchase probabilities p_1, p_2 , and expected utilities U_H, U_L are summarized by Table B.1:*

Table B.1 Customers' Purchase Probabilities and Expected Utilities

Case	Search cost c	Purchase Probabilities p_1, p_2	Expected utilities U_H, U_L
2. $v_H + v_0 < 2v_L$:			
2.1	$c \geq \gamma(v_H - v_0)$	$p_1 = 1$ $p_2 = 0$	$U_H = \gamma v_H + (1 - \gamma)v_0$ $U_L = \gamma v_L + (1 - \gamma)v_0$
2.2	$\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$	$p_1 = 1$ $p_2 = (1 - \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)v_0$ $U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]$
2.3	$\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 - \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c]$ $U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]$
2.4	$c < \gamma(v_H - v_L)$	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c]$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - c$

Proof of Proposition B.1

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1. in the proof of Proposition 1.

Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$. This is the same as Case 2.2. in the proof of Proposition 1.

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$.

• If product H is top-ranked, then the customer will search if and only if the realized value of the first product is v_0 . Hence, we have:

$$p_1 = \gamma + \frac{(1 - \gamma)^2}{2} = \frac{1 + \gamma^2}{2}, \quad U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c].$$

• If product L is top-ranked, if the realized value of the first product is v_0 , then the customer will conduct search; if the realized value of the first product is v_L , then the customer will not conduct search, in which case, we have:

$$p_2 = (1 - \gamma) \left(\gamma + \frac{1 - \gamma}{2} \right) = \frac{1 - \gamma^2}{2}, \quad U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c].$$

Case 2.4. $c < \gamma(v_H - v_L)$. This is the same as Case 1.4. in the proof of Proposition 1. $\square$

THEOREM B.1. *Sales-based ranking improves customer welfare relative to random ranking, i.e., $W_S \geq W_R$. In particular, customer welfare is strictly improved ($W_S > W_R$) if and only if $c \in (0, \gamma(v_H - v_L)) \cup (\gamma(v_L - v_0), \gamma(v_H - v_0))$.*

Proof of Theorem B.1

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1. in the proof of Theorem 1.

Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$. This is similar as Case 2.1. Specifically, based on (A.5), we know that

$$\begin{aligned} W_S > W_R &\Leftrightarrow \gamma^2 v_H + \gamma(1 - \gamma)v_0 - \gamma v_L + (1 - \gamma)c \\ &\geq \gamma[\gamma v_H + (1 - \gamma)v_0 - v_L + (1 - \gamma)(v_L - v_0)] \quad \because c \geq \gamma(v_L - v_0) \\ &= \gamma^2(v_H - v_L) > 0. \end{aligned}$$

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$.

For $s = \bar{0}$, the random walk drifts to $+\infty$ with probability

$$P(\bar{0}) = 1 + \frac{p_1[2(1 - p_2) - 1](p_1 - 1)}{2p_1 + 2(1 - p_2) - 4p_1(1 - p_2) + 2p_1(1 - p_2)^2 + 2p_1^2(1 - p_2) - p_1^2 - (1 - p_2)^2 - 1} = 1 - \frac{1 - \gamma^4}{2(1 + \gamma^4)}.$$

For $s = \underline{0}$, the random walk drifts to $+\infty$ with probability $P(\underline{0}) = 1 - P(\bar{0}) = \frac{1 - \gamma^4}{2(1 + \gamma^4)}$. Hence, we have:

$$W_R = W_S = \frac{1}{2} \{ \gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] \}.$$

Case 2.4. $c < \gamma(v_H - v_L)$. This is the same as Case 1.4. in the proof of Theorem 1. $\square$

PROPOSITION B.2. *Under $p_1 + p_2 \geq 1$, $p_1 \geq p_2$, and $p_1 > 1/2$, there exist unique $\bar{C}$ and $\underline{C}$ with $\bar{C} \geq \underline{C} \geq 0$ such that*

- (i) *If $c_B < \underline{C}$, then $(\beta_H, \beta_L) = (1, 1)$ is the unique equilibrium;*
 - (ii) *If $c_B > \bar{C}$, then $(\beta_H, \beta_L) = (0, 0)$ is the unique equilibrium;*
 - (iii) *If $\underline{C} < c_B < \bar{C}$, then $(\beta_H, \beta_L) \in (0, 1)^2$ is the unique (mixed-strategy) equilibrium; additionally, β_H (β_L) is decreasing (increasing) in c_B and there exists $\tilde{C} \in (\underline{C}, \bar{C})$ such that $\beta_H > \beta_L$ if and only if $c_B < \tilde{C}$;*
 - (iv) *If $c_B = \underline{C}$, there are two pure-strategy equilibria: $(\beta_H, \beta_L) = (1, 1)$ and $(\beta_H, \beta_L) = (1, 0)$;*
 - (v) *If $c_B = \bar{C}$, there are two pure-strategy equilibria: $(\beta_H, \beta_L) = (0, 0)$ and $(\beta_H, \beta_L) = (0, 1)$.*
- Moreover, $\bar{C}$ and $\underline{C}$ both increasing in c and (1) $\bar{C} = \underline{C} = 0$ if $c \leq \gamma(v_H - v_L)$; (2) $\bar{C} = \underline{C} > 0$ if $c \in (\gamma(v_H - v_L), \gamma(v_L - v_0)) \cup [\gamma(v_H - v_0), +\infty)$; (3) $\bar{C} > \underline{C} > 0$ if $\gamma(v_L - v_0) < c < \gamma(v_H - v_0)$.*

Proof of Proposition B.2

For (i)-(v), the proof is exactly the same as the proof of Proposition 2.

To show $\bar{C}$ and $\underline{C}$ both increasing in c and (1) $\bar{C} = \underline{C} = 0$ if $c \leq \gamma(v_H - v_L)$; (2) $\bar{C} = \underline{C} > 0$ if $c \in (\gamma(v_H - v_L), \gamma(v_L - v_0)) \cup [\gamma(v_H - v_0), +\infty)$; (3) $\bar{C} > \underline{C} > 0$ if $\gamma(v_L - v_0) < c < \gamma(v_H - v_0)$, it also follows similar derivation as in the proof of Proposition 2. Note that the expressions of $\bar{C}$ and $\underline{C}$ are given by (A.17).

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1. in the proof of Proposition 2.

Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$. This is the same as Case 1.2. in the proof of Proposition 2. Hence, it is clear that $\overline{C}(c_1) > \overline{C}(c_2)$ and $\underline{C}(c_1) > \underline{C}(c_2)$, where $c_1 \geq \gamma(v_H - v_0)$ and $\gamma(v_L - v_0) \leq c_2 < \gamma(v_H - v_0)$.

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$. In this case, we have $p_1 = \frac{1+\gamma^2}{2} = 1 - p_2$ and $p_2 = \frac{1-\gamma^2}{2}$. It is easy to verify that $\overline{C} = \underline{C} > 0$ in this case, as $r_2 = 1/r_3$, which gives

$$\begin{aligned}\underline{C}(c_3) &= \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 + \frac{\delta[(1 - p_1)(r_3 - \delta)(2r_2^Q - 1) + p_2 r_3(1 - \delta r_2)]}{(\delta p_1 r_2 - 1)(\delta(p_2 - 1) + r_3) + \delta^2 p_2 r_2(1 - p_1)} \right) \\ &= \frac{R(p_1 - p_2)}{2Q(1 - \delta)} \left(1 - \frac{\delta p_2(r_3 - \delta)2r_3^{1-Q}}{(\delta(p_2 - 1) + r_3)^2 - \delta^2 p_2^2} \right), \quad \because (A.12),\end{aligned}$$

where $\gamma(v_H - v_L) \leq c_3 < \gamma(v_L - v_0)$.

We need to show $\underline{C}(c_2) > \underline{C}(c_3)$, where $\gamma(v_L - v_0) \leq c_2 < \gamma(v_H - v_0)$ and

$$\underline{C}(c_2) = \frac{R(1 - p_2)}{2(1 - \delta)Q} \left(1 - \frac{\delta p_2}{\delta(p_2 - 1)/r_3 + 1} \right).$$

Note that to show $\underline{C}(c_2) > \underline{C}(c_3)$, it suffices to show

$$\begin{aligned}(1 - p_2) \left(1 - \frac{\delta p_2 r_3}{\delta(p_2 - 1) + r_3} \right) &> p_1 - p_2 \\ \Leftrightarrow p_2 &> \frac{\delta p_2(1 - p_2)r_3}{\delta(p_2 - 1) + r_3} \\ \Leftrightarrow \delta(p_2 - 1) + r_3 &> \delta(1 - p_2)r_3 \\ \Leftrightarrow r_3 &> \delta(1 - p_2)(r_3 - \delta),\end{aligned}$$

which is obviously true.

Case 2.4. $c < \gamma(v_H - v_L)$. This is the same as Case 1.4. in the proof of Proposition 2. $\square$

THEOREM B.2.

(i) When $c \in (0, \gamma(v_L - v_0)] \cup [\gamma(v_H - v_0), +\infty]$, then $W_B = W_S$;

(ii) When $c \in (\gamma(v_L - v_0), \gamma(v_H - v_0))$, then $W_B = W_S$ for $c_B \in (0, \underline{C}) \cup (\overline{C}, +\infty)$, where $\underline{C}$ and $\overline{C}$ are defined in Proposition 2 and W_B decreases in c_B for $c_B \in (\underline{C}, \overline{C})$. In addition, there exists $\overline{C}'$ and $\underline{C}'$ such that $\overline{C} \geq \overline{C}' > \underline{C}' > \underline{C}$ and

(a) $W_B > W_S > W_R$ if and only if $\underline{C} < c_B < \underline{C}'$;

(b) $W_S > W_B > W_R$ if and only if $\underline{C}' < c_B < \overline{C}'$.

(c) $W_S > W_R > W_B$ if and only if $\overline{C}' < c_B < \overline{C}$.

Note that there exists $\bar{\gamma} \in (0, 1)$ such that $\overline{C}' < \overline{C}$ if and only if $\gamma > \bar{\gamma}$.

Proof of Theorem B.2

Part (i).

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1 in the proof of Theorem 2.

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$.

In this case, no mixed strategy equilibrium exists, as $\underline{C} = \overline{C}$, which has been shown in the proof of Proposition B.2. Hence, brushing

$$W_B = W_S = \frac{1}{2} \{ \gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] \}.$$

Case 2.4. $c < \gamma(v_H - v_L)$. This is the same as Case 1.4. in the proof of Theorem 2.

Part (ii).

The proof is exactly the same as the proof in Theorem 2 and we can obtain $\overline{C}'_2$ and $\underline{C}'_2$. $\square$

THEOREM B.3. *As search cost c decreases,*

- (i) *Customer welfare under random ranking, W_R , and that under sales-based ranking, W_S , both increase;*
- (ii) *Customer welfare under brushing, $W_B(c)$, can decrease;*

in particular, there exists $\bar{\gamma}' \in (0, 1)$ such that $\overline{C}' < \overline{C}$ defined in Theorem 2 and for $c_B \in (\overline{C}', \overline{C})$, we have $W_B(c) < W_B(c')$, $\forall \gamma \in (\bar{\gamma}', 1)$ and $\forall c < c'$, where $c \in (\gamma(v_L - v_0), \gamma(v_H - v_0))$ and $c' > \gamma(v_H - v_0)$.

Proof of Theorem B.3

Part (i).

Based on the proof of Theorem B.1, we know that:

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Let $c_1 \geq \gamma(v_H - v_0)$, $\gamma(v_L - v_0) \leq c_2 < \gamma(v_H - v_0)$, $\gamma(v_H - v_L) \leq c_3 < \gamma(v_L - v_0)$ and $c_4 < \gamma(v_H - v_L)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1. in the proof of Theorem 3.

Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$. This is the same as Case 2.1. in the proof of Theorem 3.

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$.

$$W_R = W_S = \frac{1}{2} [\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]].$$

We have $W_R(c_3) = W_R(c_2)$ and

$$W_S(c_3) > W_S(c_2)$$

$$\begin{aligned} &\Leftrightarrow \frac{1 - \gamma^2}{1 + \gamma^2} [\gamma v_H + (1 - \gamma)v_0] + \frac{2\gamma^2}{1 + \gamma^2} \{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\} > \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] \\ &\Leftrightarrow \gamma v_H + (1 - \gamma)v_L - \gamma v_L - (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c] \Leftrightarrow \gamma(v_H - v_L) + (1 - \gamma)^2(v_L - v_0) > 0. \end{aligned}$$

Case 2.4. $c < \gamma(v_H - v_L)$.

We have:

$$\begin{aligned} W_R(c_4) > W_R(c_3) &\Leftrightarrow (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c] + \gamma[\gamma v_H + (1 - \gamma)v_L] - c > (1 - \gamma)v_0 + \gamma v_L - c(1 - \gamma) \\ &\Leftrightarrow (1 - \gamma)[\gamma(v_L - v_0) - c] + \gamma[\gamma(v_H - v_L) - c] > 0, \end{aligned}$$

since $\gamma(v_L - v_0) > \gamma(v_H - v_L) > c$.

We also have

$$\begin{aligned} W_S(c_4) > W_S(c_3) &\Leftrightarrow \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0 - c] > (1 - \gamma)v_0 + \gamma v_L \\ &\Leftrightarrow \gamma(v_H - v_L) + (1 - \gamma)[\gamma(v_L - v_0) - c] > 0, \end{aligned}$$

since $\gamma(v_L - v_0) > \gamma(v_H - v_L) > c$.

Part (ii).

Case 2. $v_H + v_0 < 2v_L$. In this case, we have $\gamma(v_H - v_L) \leq \gamma(v_L - v_0) < \gamma(v_H - v_0)$.

Case 2.1. $c \geq \gamma(v_H - v_0)$. This is the same as Case 1.1. in the proof of Theorem 3.

Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$.

$$W_B = \frac{1}{2}[(1 - \beta_L)(1 - \beta_H) + \beta_H\beta_L][P(\bar{0})U_H + (1 - P(\bar{0}))U_L + P(\underline{0})U_H + (1 - P(\underline{0}))U_L] \\ + \beta_H(1 - \beta_L)[P(Q)U_H + (1 - P(Q))U_L] + \beta_L(1 - \beta_H)[P(-Q)U_H + (1 - P(-Q))U_L],$$

where

$$\begin{cases} U_H = \gamma v_H + (1 - \gamma)v_0, \\ U_L = \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c], \end{cases}, \quad P(\bar{0}) = P(Q) = 1, \quad P(\underline{0}) = \frac{1 - \gamma^2}{1 + \gamma^2}, \quad P(-Q) = \left(\frac{1 - \gamma^2}{1 + \gamma^2}\right)^{Q+1},$$

as $P(-Q) = \left(\frac{1 - \gamma^2}{1 + \gamma^2}\right) \left(\frac{1 - p_2}{p_2}\right)^{-Q}$ where $p_2 = \frac{1 - \gamma^2}{2}$.

Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$.

$$W_B = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0 + \gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]].$$

Case 2.4. $c < \gamma(v_H - v_L)$. This is the same as Case 1.4. in the proof of Theorem 3.

It is clear that when $c < \gamma(v_L - v_0)$, W_B is decreasing in c . Let $c' \geq \gamma(v_H - v_0)$ and $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$. Now we show that there exists $\bar{C}'_2 \in (\underline{C}, \bar{C})$ and $\bar{\gamma}'_2 \in (0, 1)$ such that $\forall c_B \in (\bar{C}'_2, \bar{C})$ and $\forall \gamma \in (\bar{\gamma}', 1)$, we have $W(c') > W(c)$, where $\underline{C}$ and $\bar{C}$ are defined in Proposition 2.

When $c_B = \bar{C}$, then $\beta_H = 0, \beta_L = 1$, so we have:

$$W_B(c') = \frac{1}{2}[\gamma v_H + (1 - \gamma)v_0] + \frac{1}{2}[\gamma v_L + (1 - \gamma)v_0], \\ W_B(c) = P(-Q)[\gamma v_H + (1 - \gamma)v_0] + [1 - P(-Q)]\{\gamma v_L + (1 - \gamma)[\gamma v_H + (1 - \gamma)v_0 - c]\}.$$

Then we have

$$W_B(c') - W_B(c) = [\gamma v_H + (1 - \gamma)v_0][1/2 - P(-Q)] + \gamma v_L[P(-Q) - 1/2] \\ + (1 - \gamma)\{v_0/2 - [1 - P(-Q)][\gamma v_H + (1 - \gamma)v_0 - c]\} \\ = [1/2 - P(-Q)][\gamma v_H + (1 - \gamma)v_0 - \gamma v_L] + (1 - \gamma)\{v_0/2 - [1 - P(-Q)][\gamma v_H + (1 - \gamma)v_0 - c]\}.$$

It is clear that when $\gamma = 1$, the above expression is strictly positive. Hence, by continuity, there exist $\bar{\gamma}'_2 \in (0, 1)$ and $\bar{C}'_2 \in (\underline{C}, \bar{C})$ such that $\forall c_B \in (\bar{C}'_2, \bar{C})$ and $\forall \gamma \in (\bar{\gamma}'_2, 1)$, we have $W_B(c') > W_B(c)$. $\square$

Appendix C: More Details on Heterogeneous Search Costs

PROPOSITION C.1. *When an α fraction of customers have zero search costs, customers' purchase probabilities p_1, p_2 , and expected utilities U_H, U_L are summarized by Table C.1:*

Table C.1 Customers' Purchase Probabilities and Expected Utilities

Case	Purchase Probabilities p_1, p_2	Expected utilities U_H, U_L
1.1	$p_1 = 1 - \alpha(1 - \gamma^2)/2$ $p_2 = \alpha(1 + \gamma^2)/2$	$U_H = \gamma v_H + \alpha\gamma(1 - \gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$ $U_L = \alpha\gamma v_H + \gamma(1 - \alpha\gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$
1.2	$p_1 = 1 - \alpha(1 - \gamma^2)/2$ $p_2 = (1 - (1 - 2\alpha)\gamma^2)/2$	$U_H = \gamma v_H + \alpha\gamma(1 - \gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$ $U_L = \gamma(1 - \gamma + \alpha\gamma)v_H + \gamma(1 - \alpha\gamma)v_L + (1 - \gamma)^2 v_0 - (1 - \alpha)(1 - \gamma)c$
1.3	$p_1 = 1 - \alpha(1 - \gamma^2)/2$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + \alpha\gamma(1 - \gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - (1 - \alpha)c$
1.4	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0] - (1 - \alpha)(1 - \gamma)c$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - (1 - \alpha)c$
2.1	$p_1 = 1 - \alpha(1 - \gamma^2)/2$ $p_2 = \alpha(1 + \gamma^2)/2$	$U_H = \gamma v_H + \alpha\gamma(1 - \gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$ $U_L = \alpha\gamma v_H + \gamma(1 - \alpha\gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$
2.2	$p_1 = 1 - \alpha(1 - \gamma^2)/2$ $p_2 = (1 - (1 - 2\alpha)\gamma^2)/2$	$U_H = \gamma v_H + \alpha\gamma(1 - \gamma)v_L + (1 - \gamma)(1 - \alpha\gamma)v_0$ $U_L = \gamma(1 - \gamma + \alpha\gamma)v_H + \gamma(1 - \alpha\gamma)v_L + (1 - \gamma)^2 v_0 - (1 - \alpha)(1 - \gamma)c$
2.3	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 - (1 - 2\alpha)\gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0] - (1 - \alpha)(1 - \gamma)c$ $U_L = \gamma(1 - \gamma + \alpha\gamma)v_H + \gamma(1 - \alpha\gamma)v_L + (1 - \gamma)^2 v_0 - (1 - \alpha)(1 - \gamma)c$
2.4	$p_1 = (1 + \gamma^2)/2$ $p_2 = (1 + \gamma^2)/2$	$U_H = \gamma v_H + (1 - \gamma)[\gamma v_L + (1 - \gamma)v_0] - (1 - \alpha)(1 - \gamma)c$ $U_L = \gamma[v_H + (1 - \gamma)v_L] + (1 - \gamma)^2 v_0 - (1 - \alpha)c$

Case 1: $\mathbf{v}_H + \mathbf{v}_0 \geq 2\mathbf{v}_L$; 1.1: $c \geq \gamma(v_H - v_0)$; 1.2: $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$; 1.3: $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_L)$; 1.4: $c < \gamma(v_L - v_0)$. Case 2: $\mathbf{v}_H + \mathbf{v}_0 < 2\mathbf{v}_L$; 2.1: $c \geq \gamma(v_H - v_0)$; 2.2: $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$; 2.3: $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$; 2.4: $c < \gamma(v_H - v_L)$.

Proof of Proposition C.1

The informed customers are not affected by product rankings. The uninformed ones' decisions are still the same as those in Table 2. Thus, the purchase probabilities and expected utilities in Table C.1 are a convex combination of those for informed customers and those for uninformed ones shown in Tables 3 and B.1. $\square$

THEOREM C.1. *When $\alpha \in (0, 1)$, sales-based ranking improves customer welfare relative to random ranking, i.e., $W_S > W_R$.*

Proof of Theorem C.1

When $\alpha \in (0, 1)$, we have $p_1 + p_2 > 1$ and $p_1 < 1$. Therefore, we only need to consider the value of α that induces $p_2 \geq 1/2$ or $p_2 < 1/2$. Note that if $p_2 \geq 1/2$, then $P(s) \equiv 1$ by Lemma 1 and hence $W_S = U_H > W_R = (U_H + U_L)/2$. If $p_2 < 1/2$, then based on (A.2), we have:

$$P(s) = \begin{cases} 1 + \frac{p_1(1-2p_2)(p_1-1)}{2p_1+2(1-p_2)-4p_1(1-p_2)+2p_1(1-p_2)^2+2p_1^2(1-p_2)-p_1^2-(1-p_2)^2-1} \left(\frac{1-p_1}{p_1}\right)^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+ \\ \frac{(1-p_2)(2p_1-1)p_2}{2p_1+2(1-p_2)-4p_1(1-p_2)+2p_1(1-p_2)^2+2p_1^2(1-p_2)-p_1^2-(1-p_2)^2-1} \left(\frac{1-p_2}{p_2}\right)^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases}$$

We define $a^{\bar{0}} = a^{\underline{0}} \triangleq 1$, $\forall a \neq 0$. Hence,

$$W_R = \frac{1}{2}(U_H + U_L)$$

$$W_S = \frac{1}{2}\{U_H P(\bar{0}) + U_L[1 - P(\bar{0})] + U_H P(\underline{0}) + U_L[1 - P(\underline{0})]\}.$$

Note that we have:

$$W_S > W_R \Leftrightarrow U_H[P(\bar{0}) + P(\underline{0})] + U_L[2 - P(\bar{0}) - P(\underline{0})] > U_H + U_L$$

$$\begin{aligned}
&\Leftrightarrow U_H[P(\bar{0}) + P(\underline{0}) - 1] > U_L[P(\bar{0}) + P(\underline{0}) - 1] \Leftrightarrow P(\bar{0}) + P(\underline{0}) > 1 \\
&\Leftrightarrow \frac{(1-p_2)(2p_1-1)p_2 + p_1(1-2p_2)(p_1-1)}{2p_1 + 2(1-p_2) - 4p_1(1-p_2) + 2p_1(1-p_2)^2 + 2p_1^2(1-p_2) - p_1^2 - (1-p_2)^2 - 1} > 0 \\
&\Leftrightarrow (1-p_2)(2p_1-1)p_2 + p_1(1-2p_2)(p_1-1) > 0, \quad \because (A.3) \\
&\Leftrightarrow (p_1 + p_2 - 1)[p_1(1-p_2) + p_2(1-p_1)] > 0, \quad \because p_1 + p_2 > 1. \quad \square
\end{aligned}$$

THEOREM C.2. When $\alpha \in (0, 1)$, Customer welfare under the three schemes, W_R , W_S , and W_B , compare as follows: for $c_B \in (\underline{C}, \bar{C})$ defined in Proposition 2, there exists $(\underline{C}'_1, \bar{C}'_1) \in (\underline{C}, \bar{C})$ and $\bar{\gamma}_1 \in (0, 1)$ such that $W_B > W_S > W_R$ when $c_B < \underline{C}'_1$; $W_B < W_R < W_S$ when $c_B > \bar{C}'_1$ and $\gamma > \bar{\gamma}_1$ if either one of the following conditions hold:

- (i) $0 < \alpha < 1/2$ and $c \geq \gamma(v_H - v_L)$;
- (ii) $0 < \alpha < (1 + \gamma^2)^{-1}$ and $c \geq \gamma(v_H - v_0)$;

Proof of Theorem C.2

Note that $W_S > W_R$ for all $\alpha \in (0, 1)$ has been shown in Theorem C.1.

When $\alpha \in (0, 1)$, we have $p_1 + p_2 > 1$ and $p_1 < 1$. Therefore, we only need to consider the value of α that induces $p_2 < 1/2$. This is because when $p_2 \geq 1/2$, $P(s) \equiv 1$ by Lemma 1 and hence $W_B = W_S = U_H > W_R = (U_H + U_L)/2$. When $p_2 < 1/2$, based on (A.2), we have:

$$P(s) = \begin{cases} 1 + \frac{p_1(1-2p_2)(p_1-1)}{2p_1 + 2(1-p_2) - 4p_1(1-p_2) + 2p_1(1-p_2)^2 + 2p_1^2(1-p_2) - p_1^2 - (1-p_2)^2 - 1} \left(\frac{1-p_1}{p_1}\right)^s, & \text{if } s \in \{\bar{0}\} \cup \mathbb{Z}^+ \\ \frac{(1-p_2)(2p_1-1)p_2}{2p_1 + 2(1-p_2) - 4p_1(1-p_2) + 2p_1(1-p_2)^2 + 2p_1^2(1-p_2) - p_1^2 - (1-p_2)^2 - 1} \left(\frac{1-p_2}{p_2}\right)^s, & \text{if } s \in \{\underline{0}\} \cup \mathbb{Z}^-. \end{cases}$$

We also have

$$\begin{aligned}
W_R &= \frac{1}{2}(U_H + U_L) \\
W_S &= \frac{1}{2}\{U_H P(\bar{0}) + U_L[1 - P(\bar{0})] + U_H P(\underline{0}) + U_L[1 - P(\underline{0})]\} \\
W_B &= \frac{1}{2}[(1 - \beta_L)(1 - \beta_H) + \beta_H \beta_L][P(\bar{0})U_H + (1 - P(\bar{0}))U_L + P(\underline{0})U_H + (1 - P(\underline{0}))U_L] \\
&\quad + \beta_H(1 - \beta_L)[P(Q)U_H + (1 - P(Q))U_L] + \beta_L(1 - \beta_H)[P(-Q)U_H + (1 - P(-Q))U_L].
\end{aligned}$$

The following cases listed in Table C.1 can produce $p_2 < 1/2$ subject to the conditions imposed on α and γ :

- Case 1.1. $c \geq \gamma(v_H - v_0)$: $p_2 < 1/2$ if and only if $\alpha < (1 + \gamma^2)^{-1}$.
- Case 1.2. $\gamma(v_H - v_L) \leq c < \gamma(v_H - v_0)$: $p_2 < 1/2$ if and only if $\alpha < 1/2$.
- Case 2.1. $c \geq \gamma(v_H - v_0)$: $p_2 < 1/2$ if and only if $\alpha < (1 + \gamma^2)^{-1}$.
- Case 2.2. $\gamma(v_L - v_0) \leq c < \gamma(v_H - v_0)$: $p_2 < 1/2$ if and only if $\alpha < 1/2$.
- Case 2.3. $\gamma(v_H - v_L) \leq c < \gamma(v_L - v_0)$: $p_2 < 1/2$ if and only if $\alpha < 1/2$.

Conditions for $W_B > W_S$ follow from the same argument in the proof of Theorem 2-(i).

To derive conditions for $W_B < W_R$, from the proof in the part (ii) of Theorem 2, we know from (A.19) that it suffices to derive the condition on γ such that $1/2 > P(-Q)$, which is equivalent to

$$\frac{(1-p_2)(2p_1-1)p_2}{2p_1 + 2(1-p_2) - 4p_1(1-p_2) + 2p_1(1-p_2)^2 + 2p_1^2(1-p_2) - p_1^2 - (1-p_2)^2 - 1} \left(\frac{1-p_2}{p_2}\right)^{-Q} < 1/2. \quad (C.1)$$

As p_1, p_2 are functions of γ , to derive conditions on γ such that (C.1) holds, it suffices to derive conditions on γ such that the following inequality holds

$$\frac{(1-p_2)(2p_1-1)p_2}{2p_1+2(1-p_2)-4p_1(1-p_2)+2p_1(1-p_2)^2+2p_1^2(1-p_2)-p_1^2-(1-p_2)^2-1} < 1/2,$$

which is equivalent to

$$2p_1^2p_2 + p_1^2 + 2p_1p_2^2 - 6p_1p_2 - p_2^2 + 2p_2 > 0. \quad (\text{C.2})$$

Note that for Case 1.1, 1.2, 2.1, 2.2 and 2.3, we need to consider the following combinations of (p_1, p_2) .

$$\begin{aligned} \text{Case 1.1 and 2.1.} & \begin{cases} p_1 = 1 - \alpha + \frac{\alpha(1+\gamma^2)}{2}, \\ p_2 = \frac{\alpha(1+\gamma^2)}{2}. \end{cases} \\ \text{Case 1.2 and 2.2.} & \begin{cases} p_1 = 1 - \alpha + \frac{\alpha(1+\gamma^2)}{2}, \\ p_2 = \frac{(1-\alpha)(1-\gamma^2)}{2} + \frac{\alpha(1+\gamma^2)}{2}. \end{cases} \\ \text{Case 2.3.} & \begin{cases} p_1 = \frac{(1-\alpha)(1+\gamma^2)}{2} + \frac{\alpha(1+\gamma^2)}{2}, \\ p_2 = \frac{(1-\alpha)(1-\gamma^2)}{2} + \frac{\alpha(1+\gamma^2)}{2}. \end{cases} \end{aligned} \quad (\text{C.3})$$

When $\gamma = 1$, p_1 and p_2 in (C.3) are equal to α . Therefore, when $\gamma = 1$, (C.2) becomes

$$2\alpha + 1 + 2\alpha^2 - 6\alpha - \alpha^2 + 2\alpha > 0 \Leftrightarrow (\alpha - 1)^2 > 0,$$

which is obviously true for $\alpha \in (0, 1)$. Therefore, by continuity, there exists $\bar{\gamma}_1$ such that for $\gamma > \bar{\gamma}_1$, the claim of this theorem is true. $\square$